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Variance-reduced weak Monte Carlo simulations of stochastically driven oscillators of engineering interest

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Abstract

Analytical solutions of nonlinear and higher-dimensional stochastically driven oscillators are rarely possible and this leaves the direct Monte Carlo simulation of the governing stochastic differential equations (SDEs) as the only tool to obtain the required numerical solution. Engineers, in particular, are mostly interested in weak numerical solutions, which provide a faster and simpler computational framework to obtain the statistical expectations (moments) of the response functions. The numerical integration tools considered in this study are weak versions of stochastic Euler and stochastic Newmark methods. A well-known limitation of a Monte Carlo approach is however the requirement of a large ensemble size in order to arrive at convergent estimates of the statistical quantities of interest. Presently, a simple form of a variance reduction strategy is proposed such that the ensemble size may be significantly reduced without affecting the accuracy of the predicted expectations of any function of the response vector process provided that the function can be adequately represented through a power-series approximation. The basis of the variance reduction strategy is to appropriately augment the governing system equations and then weakly replace the stochastic forcing function (which is typically a filtered white noise process) through a set of variance-reduced functions. In the process, the additional computational cost due to system augmentation is far smaller than the accrued advantages due to a drastically reduced ensemble size. Indeed, we show that the proposed method appears to work satisfactorily even in the special case of the ensemble size being just 1. The variance reduction scheme is first illustrated through applications to a nonlinear Duffing equation driven by additive and multiplicative white noise processes—a problem for which exact stationary solutions are known. This is followed up with applications of the strategy to a few higher-dimensional systems, i.e., 2- and 3-dof nonlinear oscillators under additive white noises.

1. Introduction

There are quite a few approximate analytical techniques to determine the probability distributions or first few statistical moments of (generally low dimensional) nonlinear oscillators driven by stochastic excitations, often modelled via white noise processes. The equivalent linearization [1,2], higher-order linearization [3], equivalent nonlinearization [4], perturbation [5], moment closures [6], stochastic averaging [7–9], conditional linearization [10], phase space linearization [11] are just to name a few. Unfortunately, these methods have generally not been applicable to nonlinear oscillators with large degrees-of-freedom (dof) and when driven by

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an arbitrary combination of additive and multiplicative noises. A far more potential route is to pursue a Monte Carlo simulation (MCS) and directly integrate the equations of motion, which are generally in the form of stochastic differential equations (SDEs). Direct integration schemes for SDEs may be broadly categorized as strong (or pathwise) [12] and weak [13–16]. Several strong integration schemes, such as Euler, Heun, Milstein, local linearization [11,17–19] schemes, are presently available. Unlike strong solutions, a weak solution has no pathwise resemblance with a realization of the exact solution. However, the mathematical expectation of an arbitrary deterministic function of a weak solution is close to that of the exact solution. Such a closeness is defined in terms of a certain order of the given time step size [16]. Unlike strong approaches, a weak approach is simpler and computationally faster. Recently, weak integration techniques using quasi-random numbers have also been proposed ([20]). Unlike many of the highly accurate integration schemes for deterministic oscillators, stochastic integration schemes almost always have a significantly lower order of accuracy and accordingly the accuracy of MCS-based simulations of statistical moments is limited too.

Yet another source of inaccuracy is due to the finiteness of the ensemble size, which is the number of samples (simulated trajectories) N constituting the ensemble. The statistical error in a crude simulation is merely proportional to $1/\sqrt{N}$ and thus it may take an enormous ensemble size to reduce this error to acceptable limits. A variance reduction procedure however enables the computation of the statistical expectations possibly to the same order of accuracy as permitted by the numerical integration scheme while drastically reducing N [16,21]. The most popularly known existing theory on variance reduction is based on a Girsanov's transformation of the probability measure [16,22,27], a concept that draws heavily on stochastic calculus and control while being able to reduce the variance of the expectation of only a single given function of the response process that is differentiable with respect to the initial conditions. Moreover, a numerical implementation of such an approach is quite difficult, as it requires an initial guess of the zero-variance expectation as well as its derivatives with respect to initial conditions to be found. In the present study, we intend to explore a host of weak numerical integration strategies for fast and accurate MCS-based simulations of nonlinear mechanical oscillators of interest in structural dynamics. In particular, we explore the weak forms of explicit and implicit stochastic Euler methods and the weak stochastic Newmark method (SNM). The objective is to obtain the statistical moment histories of deterministic functions of response vector processes. We also attempt to derive a somewhat novel variance reduction approach that requires no educated guesses on the statistical moments of interest and that can simultaneously handle the variance reduction of expectations of any set of functions provided that such functions can be approximated via Taylor- or power-series expansions. The essence of this approach is to suitably augment the given system of governing equations (SDEs) by additional SDEs for appropriate powers of the response vector functions (such as displacement and velocity vector functions). These additional equations are naturally driven by stochastic forcing functions that are appropriate powers of the white noise processes (i.e., formal derivatives of Wiener processes) driving the original SDEs. The reduction of variance is achieved through a weak replacement of the augmented set of stochastic forcing processes with known statistical properties by a set of variance-reduced forcing processes. The method can treat nonlinear oscillators of large dimensions with additive and multiplicative stochastic inputs. Moreover, it may be applied for a deterministic, direct simulation of the required expectation with an ensemble size of just 1. We illustrate the proposed strategies to a limited extent through their applications to a hardening Duffing equation driven by additive and multiplicative white noise processes.

2. Method of analysis

Consider the following form of an n-dof dynamical system used mostly in the context of engineering dynamics:

$$[M]\ddot{X} + C(X,\dot{X})\dot{X} + K(X,\dot{X})X = \sum_{r=1}^{q} G_r(X,\dot{X},t)\dot{W}_r + F(t), \tag{1}$$

where $X = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}^{\mathrm{T}} \in \mathbf{R}^n$ is the displacement vector, $[M]$ is a constant mass matrix, $C(X,\dot{X})$, $K(X,\dot{X}) \in \mathbf{R}^{n\times n}$ are (state dependent for nonlinear systems) damping and stiffness matrices, respectively, $\{G_r(X,\dot{X},t) : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n\}$ is the rth element of a set of n diffusion vectors, $\{W_r(t)|r \in [1,q]\}$ constitutes a q-dimensional vector of independently evolving zero-mean Wiener processes with $W_r(0) = 0$, $\mathbf{E}[|W_r(t) - W_r(s)|^2] = (t-s), t>s\ \forall r\in[1,q]$ and $F(t) = \{F^{(j)}(t)|j = 1,\ldots,n\}$ is the externally applied (non-parametric) and deterministic force vector. $\mathbf{E}$ denotes the expectation operator. The description of the oscillating system as in Eq. (1) being entirely formal (due to non-differentiability of Wiener processes, which implies that $\dot{W}_r(t)$exists merely as a valid measure, but not as a mathematical function), they may more appropriately be recast as the following system of *2n* first-order SDEs in an incremental form:

$$\begin{aligned}
\mathrm{d}x_1^{(j)} &= x_2^{(j)}\,\mathrm{d}t,\\
\mathrm{d}x_2^{(j)} &= a^{(j)}(X,\dot{X},t)\,\mathrm{d}t + \sum_{r=1}^{q} b_r^{(j)}(X,\dot{X},t)\,\mathrm{d}W_r(t); \quad j=1,2,\ldots,n,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
a^{(j)}(X,\dot{X},t) &= -\sum_{k=1}^{n}\bar{C}_{jk}(X,\dot{X})\dot{x}^{(k)} - \sum_{k=1}^{n}\bar{K}_{jk}(X,\dot{X})x^{(k)} + \bar{F}^{(j)}(t),\\
b_r^{(j)}(X,\dot{X},t) &= \bar{G}_r^{(j)}(X,\dot{X},t),\\
[\bar{C}] = [M^{-1}][C],\quad [\bar{K}] &= [M^{-1}][K],\quad \{\bar{F}\} = [M^{-1}]\{F\},\quad \{\bar{G}\} = [M^{-1}]\{G\}.
\end{aligned} \tag{3}$$

As will be seen shortly, an explicit computation of the inverse $[M^{-1}]$ is not necessary. To ensure boundedness of the solution vector $\bar{X} \stackrel{\Delta}{=} \{X^{\mathrm{T}}, \dot{X}^{\mathrm{T}}\}^{\mathrm{T}} \in \mathbf{R}^{2n}$ (where $X \stackrel{\Delta}{=} \{x_1^{(j)}\} \in \mathbf{R}^n$ and $\dot{X} \stackrel{\Delta}{=} \{x_2^{(j)}\} \in \mathbf{R}^n, j=1,\ldots,n$), it is assumed that the drift and diffusion vectors, $\mathbf{a} = \{a^{(j)}\}$ and $\mathbf{b}_r = \{b_r^{(j)}\}$ are measurable, continuous, satisfying Lipschitz growth bound:

$$||\mathbf{a}(\bar{X},t) - \mathbf{a}(\bar{Y},t)|| + \sum_{r=1}^{q} ||\mathbf{b_r}(\bar{X},t) - \mathbf{b_r}(\bar{Y},t)|| \leqslant Q||\bar{X} - \bar{Y}||, \tag{4}$$

where $\bar{Y} \in \mathbf{R}^{2n}$, $Q \in R^+$ and $||\cdot||$ denotes the Euclidean norm (i.e., the L_2 norm in the associated probability space $[\Omega,F,P]$). Let the initial conditions be mean square bounded, i.e., $\mathbf{E}||\bar{X}(t_0)||^2 < \infty$ (for most practical purposes the initial condition vector may be treated as deterministic and this is followed in this study too). The time interval $[0,T]$ of interest is so ordered that $0 = t_0 < t_1 \cdots < t_i < \cdots < t_L = T$ and $h_i = t_i - t_{i-1}$ where $i \in Z^+$ For further simplification of the rest of the presentation, we assume a uniform time step $h_i = h \forall i$.

We intend to approximate the set of Eq. (2) by weak integration methods, viz. weak (explicit) Euler (WE) method, weak implicit Euler (WIE) method and weak stochastic Newmark method (WSNM) over the ith time interval $T_i = (t_{i-1}, t_i]$, given the initial condition vector $\bar{X}(t_{i-1}) \stackrel{\Delta}{=} \bar{X}_{i-1}$. A weak solution of a stochastic dynamical system is referred to any approximate solution such that expectations of a sufficiently general (analytic) function of the so-called 'exact' solution and that of the approximate (weak) solution agree within some order of a given time step. Since the first few moments of the 'exact' and weak solutions also match within the same order of accuracy, determination of weak solutions often suffices from the viewpoint of many engineering applications and such an approach evidently saves considerable computational costs. This means that if $\bar{X}^{(w)}(t_i) = \{X^{(w)}(t_i), \dot{X}^{(w)}(t_i)\}^{\mathrm{T}} \in \mathbf{R}^n$ denotes the approximate weak solution, then one has the following inequality for any (sufficiently smooth/analytic) function ψ:

$$||\{\mathbf{E}\psi(\bar{X}(t_i)) - \mathbf{E}\psi(\bar{X}^{(w)}(t_i)|\bar{X}(t_0)) = \bar{X}^{(w)}(t_0) \stackrel{\Delta}{=} \bar{X}_0\}|| \leqslant O(h^p). \tag{5}$$

Towards deriving such a map, we expand the vectors $X(t_i)$ and $\dot{X}(t_i)$ through stochastic Taylor expansions about $X(t_{i-1}) \stackrel{\Delta}{=} X_{i-1} \stackrel{\Delta}{=} X_{1,i-1}$ and $\dot{X}(t_{i-1}) \stackrel{\Delta}{=} \dot{X}_{i-1} \stackrel{\Delta}{=} X_{2,i-1}$, respectively. Such stochastic Taylor expansions require repeated application of Ito's formula, which, as adapted specifically for Eq. (2) is stated below:

$$\begin{aligned}
\psi(X(s),\dot{X}(s),s) = \psi(X_{1,i-1},X_{2,i-1},t_{i-1}) &+ \sum_{r=1}^{q}\int_{t_{i-1}}^{s} \Lambda_r\psi(X_1(s_1),X_2(s_1),s_1)\,\mathrm{d}W_r(s_1)\\
&+ \int_{t_{i-1}}^{s} L\psi(X_1(s_1),X_2(s_1),s_1)\,\mathrm{d}s_1.
\end{aligned} \tag{6}$$

Here, ψ is any (deterministic; sufficiently smooth; scalar or vector) function of its arguments, $s \geqslant t_{i-1}$ and the operators Λ_r and L are defined as

$$\Lambda_r \psi = \sum_{j=1}^{n} b_r^{(j)} \frac{\partial \psi(X_1, X_2, t)}{\partial x_2^{(j)}},$$
$$L\psi = \frac{\partial \psi}{\partial t} + \sum_{j=1}^{n} x_2^{(j)} \frac{\partial \psi}{\partial x_1^{(j)}} + \sum_{j=1}^{n} a^{(j)} \frac{\partial \psi}{\partial x_2^{(j)}} + 0.5 \sum_{r=1}^{q} \sum_{k=1}^{n} \sum_{l=1}^{n} b_r^{(k)} b_r^{(l)} \frac{\partial^2 \psi}{\partial x_2^{(k)} \partial x_2^{(l)}}. \tag{7}$$

Now a two-term Ito-Taylor expansion of the jth displacement (scalar) component, $x_{1,i}^{(j)} \stackrel{\Delta}{=} x_1^{(j)}(t_{i-1} + h)$ results in the strong (pathwise) form of the stochastic Euler method (provided that the remainder term is removed):

$$x_{1,i}^{(j)} = x_{1,i-1}^{(j)} + \overbrace{x_{2,i-1}^{(j)} h}^{\text{Term-I}} + R_{E,d}^{(j)}, \tag{8}$$

where the jth displacement remainder $R_{E,d}^{(j)}$, is given by

$$R_{E,d}^{(j)} = \underbrace{\sum_{r=1}^{q} b_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathrm{d}W_r(s_1)\mathrm{d}s}_{\text{Term-II}} + \underbrace{0.5 a^{(j)}(\bar{X}_{i-1}, t_{i-1}) h^2}_{\text{Term-III}}$$
$$+ \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \Lambda_r a^{(j)}(\bar{X}(s_2), s_2)\, \mathrm{d}W_r(s_2)\, \mathrm{d}s_1\, \mathrm{d}s + \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} L a^{(j)}(\bar{X}(s_2), s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1\, \mathrm{d}s$$
$$+ \sum_{r=1}^{q} \sum_{l=1}^{q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \Lambda_l b_r^{(j)}(\bar{X}(s_2), s_2)\, \mathrm{d}W_l(s_2)\, \mathrm{d}W_r(s_1)\, \mathrm{d}s +$$
$$+ \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} L b_r^{(j)}(\bar{X}(s_2), s_2)\, \mathrm{d}s_2\, \mathrm{d}W_r(s_1)\, \mathrm{d}s. \tag{9}$$

A similar exercise for the jth velocity component yields

$$x_{2,i}^{(j)} = x_{2,i-1}^{(j)} + \sum_{r=1}^{q} b_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) \int_{t_{i-1}}^{t_i} \mathrm{d}W_r + \underbrace{a^{(j)}(\bar{X}_{i-1}, t_{i-1}) h}_{\text{Term-I}} + R_{E,v}^{(j)}, \tag{10}$$

where the expression for the velocity remainder $R_{E,v}^{(j)}$ is

$$R_{E,v}^{(j)} = \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \mathrm{d}W_r(s) \int_{t_{i-1}}^{s} \sum_{l=1}^{q} \Lambda_l b_r^{(j)}(\bar{X}(s_1), s_1)\, \mathrm{d}W_l(s_1)$$
$$+ \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \mathrm{d}W_r(s) \int_{t_{i-1}}^{s} L b_r^{(j)}(\bar{X}(s_1), s_1)\, \mathrm{d}s_1. \tag{11}$$

Indeed we obtain strong stochastic Euler maps for jth displacement and velocity components provided the remainder terms $R_{E,d}^{(j)}$ and $R_{E,v}^{(j)}$ are removed from Eqs. (8) and (10), respectively. The drift-implicit versions of the strong Euler method (SEM), i.e., the IE maps, are obtained if we replace *Term-I* of the jth displacement Eq. (8) using a non-unique implicitness parameter (real scalar) 'α' by

$$x_{2,i-1}^{(j)} h = \alpha x_{2,i-1}^{(j)} h + (1 - \alpha) x_{2,i}^{(j)} h. \tag{12}$$

Hence, the remainder$R_{E,d}^{(j)}$ changes to $R_{IE,d}^{(j)}$ as

$$R_{IE,d}^{(j)} = -(1 - \alpha)\rho_1^{(j)} h + R_{E,d}^{(j)}, \tag{13}$$

where

$$\rho_1^{(j)} = \int_{t_{i-1}}^{t_i} L a^{(j)}(\bar{X}(s), s)\, \mathrm{d}s + \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \Lambda_r a^{(j)}(\bar{X}(s), s)\, \mathrm{d}W_r(s). \tag{14}$$

Similarly, we modify *Term-I* (i.e., the drift part of the velocity map) of the *j*th velocity equation (10) with another implicitness parameter (real scalar) β as

$$a^{(j)}(\bar{X}_{i-1}, t_{i-1})h = \beta a^{(j)}(\bar{X}_{i-1}, t_{i-1})h + (1-\beta)(\bar{X}_i, t_i)h. \tag{15}$$

Hence, $R^{(j)}_{E,v}$ changes to $R^{(j)}_{IE,v}$ as

$$R^{(j)}_{IE,v} = -(1-\beta)\rho^{(j)}_1 h + R^{(j)}_{E,v}. \tag{16}$$

In the strong form of the stochastic Newmark method (SNM), we obtain the *j*th displacement component by adding *Term-II* and *Term-III* of Eq. (9) to the non-remainder part of the RHS of Eq. (8) and then use an implicitness parameter 'α' to modify *Term-III*. At the end of the exercise, we arrive at the following expression:

$$\begin{aligned} x^{(j)}_{1,i} = x^{(j)}_{1,i-1} + x^{(j)}_{2,i-1}h + \sum_{r=1}^{q} \sigma^{(j)}_r(\bar{X}_{i-1}, t_{i-1}) \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s} \mathrm{d}W_r(s_1)\mathrm{d}s \\ + 0.5\alpha a^{(j)}(\bar{X}_{i-1}, t_{i-1})h^2 + 0.5(1-\alpha)a^{(j)}(\bar{X}_i, t_i)h^2 + R^{(j)}_{\mathrm{SNM},d}, \end{aligned} \tag{17}$$

where the strong displacement remainder $R^{(j)}_{\mathrm{SNM},d}$ is given by

$$\begin{aligned} R^{(j)}_{\mathrm{SNM},d} = & \sum_{r=1}^{q}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \Lambda_r a^{(j)}(\bar{X}(s_2), s_2)\,\mathrm{d}W_r(s_2)\,\mathrm{d}s_1\,\mathrm{d}s + \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} La^{(j)}(\bar{X}(s_2), s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s \\ & + \sum_{r=1}^{q}\sum_{l=1}^{q}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \Lambda_l \sigma^{(j)}_r(\bar{X}(s_2), s_2)\,\mathrm{d}W_l(s_2)\,\mathrm{d}W_r(s_1)\,\mathrm{d}s \\ & + \sum_{r=1}^{q}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} L\sigma^{(j)}_r(\bar{X}(s_2), s_2)\,\mathrm{d}s_2\,\mathrm{d}W_r(s_1)\,\mathrm{d}s - 0.5(1-\alpha)\rho^{(j)}_1. \end{aligned} \tag{18}$$

The SNM map for the *j*th displacement component is obtained by removing the remainder term from Eq. (17). The *j*th velocity component of the SNM map is the same as that of the IE map.

In strong solutions, we model the multiple stochastic integrals (MSIs) $I_r = \int_{t_i}^{t_{i-1}} \mathrm{d}W_r(s)$ (for explicit and implicit Euler maps) or I_r and $I_{r0} = \int_{t_i}^{t_{i-1}}\int_{t_{i-1}}^{s} \mathrm{d}W_r(s_1)\,\mathrm{d}s$, (for the SNM map) as zero-mean Gaussian (normal) random variables. For instance, it can be shown that that I_r and I_{r0} have variances of h and $h^3/3$, respectively. These Gaussian variables are typically generated as certain transcendental (logarithmic and/or sinusoidal) transformations (known as Box–Muller or Polar–Marsaglia transformations) of uniformly distributed random variables in [0,1]. Thus, in the strong or pathwise approaches, the variances of the MSIs would have to be found and then Gaussian realizations of these variables would be generated for determination of the MSIs involving a set of Wiener increments. However, in a weak approach, it suffices to replace the MSIs by a set of random variables corresponding to a far simpler set of probability distributions so that inequality (5) holds for some p. Let the weak replacements for $I_r = O(\sqrt{h})$ and $I_{r0} = O(h^{1.5})$ be, respectively, denoted by λ_r and η_r. The weak random variables λ_r and η_r are obtainable from probability distributions that are non-unique and one possible set of distributions is derived in Section 4, where we consider a few numerical examples. Hence, the weak maps for displacement and velocity vectors are given (based on the displacement and velocity maps after pre-multiplying both sides by $[M]$ and dropping superscripts (j)) as:

Weak Euler method (*WEM*):

$$MX_{\mathrm{WE},i} = MX_{\mathrm{WE},i-1} + M\dot{X}_{\mathrm{WE},i-1}h, \tag{19}$$

$$M\dot{X}_{\mathrm{WE},i} = M\dot{X}_{\mathrm{WE},i-1} + \sum_{r=1}^{q} G_r(\bar{X}_{\mathrm{WE},i-1}, t_{i-1})\lambda_r. \tag{20}$$

Weak implicit Euler method (*WIEM*):

$$MX_{\mathrm{WIE},i} = MX_{\mathrm{WIE},i-1} + \alpha M\dot{X}_{\mathrm{WIE},i-1}h + (1-\alpha)M\dot{X}_{\mathrm{WIE},i}h, \tag{21}$$

$$M\dot{X}_{\mathrm{WIE},i} = M\dot{X}_{\mathrm{WIE},i-1} + \sum_{r=1}^{q} G_r(\bar{X}_{i-1}, t_{i-1})\lambda_r + [\beta a(\bar{X}_{i-1}, t_{i-1}) + (1-\beta)a(\bar{X}_{L,i}, t_i)]h. \tag{22}$$

Substituting for $a(\bar{X}_{i-1}, t_{i-1})$ and $a(\bar{X}_i, t_i)$, we finally have

$$\begin{aligned} M\dot{X}_{\mathrm{WIE},i} = {} & M\dot{X}_{\mathrm{WIE},i-1} + \sum_{r=1}^{q} G_r(\bar{X}_{i-1}, t_{i-1})\lambda_r \\ & - \beta[C(\bar{X}_{i-1}, t_{i-1})\dot{X}_{i-1} + K(\bar{X}_{i-1}, t_{i-1})X_{i-1} - F(t_{i-1})]h \\ & - (1-\beta)[C(\bar{X}_i, t_i)\dot{X}_i + K(\bar{X}_i, t_i)X_i - F(t_i)]h. \end{aligned} \tag{23}$$

Weak stochastic Newmark method (*WSNM*):

$$\begin{aligned} MX_{\mathrm{WSNM},i} = {} & MX_{\mathrm{WSNM},i-1} + M\dot{X}_{\mathrm{WSNM},i-1}h + \sum_{r=1}^{q} G_r(\bar{X}_{i-1}, t_{i-1})\eta_r \\ & - 0.5\alpha[C(\bar{X}_{i-1}, t_{i-1})\dot{X}_{i-1} + K(\bar{X}_{i-1}, t_{i-1})X_{i-1} - F(t_{i-1})]h^2 \\ & - 0.5(1-\alpha)[C(\bar{X}_i, t_i)\dot{X}_i + K(\bar{X}_i, t_i)X_i - F(t_i)]h^2, \end{aligned} \tag{24}$$

$$\begin{aligned} M\dot{X}_{\mathrm{WSNM},i} = {} & M\dot{X}_{\mathrm{WSNM},i-1} + \sum_{r=1}^{q} G_r(\bar{X}_{i-1}, t_{i-1})\lambda_r \\ & - \beta[C(\bar{X}_{i-1}, t_{i-1})\dot{X}_{i-1} + K(\bar{X}_{i-1}, t_{i-1})X_{i-1} - F(t_{i-1})]h \\ & - (1-\beta)[C(\bar{X}_i, t_i)\dot{X}_i + K(\bar{X}_i, t_i)X_i - F(t_i)]h. \end{aligned} \tag{25}$$

As we have mentioned, λ_r η_r constitute of a set of zero-mean random variables, whose distributions may be non-uniquely established as functions of h. For instance, as derived in next section, we may choose distributions of these variables as

$$\mathrm{WEM}: \; P(\lambda_r = \pm\sqrt{h}) = 0.50, \tag{26}$$

$$\mathrm{WIEM}: \; P(\lambda_r = \pm\sqrt{h}) = 0.50, \tag{27}$$

$$\mathrm{WSNM}: P(\lambda_r = \pm\sqrt{h}) = 0.50, \quad \eta_r = 0.5h\lambda_r + \gamma_r h, \quad P(\gamma_r = \pm\sqrt{h/12}) = 0.50. \tag{28}$$

Moreover, these variables need to be so generated that for any $r \neq s$, λ_r and η_r must be independent. It is obvious that a generation of random variables with the above discrete distributions offers a much faster and simpler alternative to the generation of normal random variables, which require the usage of costly transcendental functions of uniformly distributed random variables in [0,1].

Starting from any deterministic initial condition $\bar{X}_0 = \{X_0, \dot{X}_0\}^{\mathrm{T}}$, the required subset of Eqs. (19)–(25) (as applicable for a specific method) may be recursively solved for $\bar{X}_i = \{X_i^{\mathrm{T}}, \dot{X}_i^{\mathrm{T}}\}^{\mathrm{T}}$, for each $i = 1,2,\ldots$, as a set of *2n* coupled nonlinear algebraic equations. Even though the ensemble-averaged quantities based on these weak methods approach those based on strong solutions (as the ensemble size tends to infinity), variances of the quantities tend to increase drastically for lower ensemble sizes. With this in perspective, a variance reduction technique is next devised such that, in addition to simulations of the statistical moments, variances of the simulated moments with lower ensemble sizes can also be reduced.

As we see in the above weak formulations, the randomness is associated with the variables $I_r = \int_{t_i}^{t_{i-1}} \mathrm{d}W_r(s)$ and $I_{r0} = \int_{t_i}^{t_{i-1}} \int_{t_{i-1}}^{s} \mathrm{d}W_r(s_1)\mathrm{d}s$. It can be shown that these variables are distributed according to $N(0,h)$ and $N(0,h^3/3)$, respectively. Suppose that we need to reduce the variances associated with only the simulated first moments (means) of displacement and velocity components. Then, from Eqs. (19) to (25), it is evident that we only need to reduce the variances associated with each of the terms containing I_r and I_{r0}, say by a factor $k>1$, and accordingly replace them with variance-reduced random variables λ_r and η_r (respectively) such that they are $N(0,h/k)$ and $N(0,h^3/3k)$. It is also clear that the variance reduction factor 'k' reduces the standard deviation of any linear function of λ_r or η_r by a factor $\sqrt{k}$. While we may hope to reduce the variance of only

$x_1^{(j)}$ and $x_2^{(j)}$ $(j \in [1,n])$ using this approach, we are still to attain our objective of reducing variance of any pth degree polynomial of the form

$$P(x_1^{(j)}, x_2^{(j)} | j \in [1,n]) = \sum_{k=0}^{p} \sum_{\sum_{i\in[1,n]} (p_i+q_i)=k} A_{p_1,\ldots,p_n,q_1,\ldots,q_n} x_1^{p_1,(1)}, \ldots, x_1^{p_n,(n)}, \ldots, x_2^{q_1,(1)}, \ldots, x_2^{q_n,(n)}, \tag{29}$$

where the coefficients $A_{p_1,\ldots,p_n,q_1,\ldots,q_n}$ are real and we have adopted the convention: $x_1^{p_j,(j)} \triangleq (x_1^{(j)})^{p_j}$, and $x_2^{q_j,(j)} \triangleq (x_2^{(j)})^{q_j}$. In doing so, we hope to be able reduce the variance of any smooth (sufficiently differentiable) function of the response process, since such a function should be reproducible (to a certain order of accuracy) using a polynomial (Taylor expansion) of the above form. Thus, to begin with, we consider the stochastic Euler method and write down equations up to the second powers (i.e., $x_1^{p_j,(j)}$ and $x_2^{q_j,(j)}$ with $0 \leqslant p_j, q_j \leqslant 2$) of the displacement and velocity components through the (explicit) Euler map (with the remainder terms removed):

$$x_{1,i}^{(j)} = x_{1,i-1}^{(j)} + x_{2,i-1}^{(j)} h + R_{E,d}^{(j)} \quad j \in [1,n], \tag{30}$$

$$x_{2,i}^{(j)} = x_{2,i-1}^{(j)} + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) I_r + a^{(j)}(\bar{X}_{i-1}, t_{i-1}) h + R_{E,v}^{(j)} \quad j \in [1,n]. \tag{31}$$

Multiplying the jth displacement map with the kth (as in Eq. (2.29)), we have the following identity:

$$x_{1,i}^{(j)} x_{1,i}^{(k)} = x_{1,i-1}^{(j)} x_{1,i-1}^{(k)} + x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} h^2 + x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} h + x_{1,i-1}^{(k)} x_{2,i-1}^{(j)} h + R_{E,d^2}^{(j,k)} \quad (j,k) \in [1,n] \times [1,n] \tag{32}$$

where $R_{E,d^2}^{(j,k)}$ is the remainder term corresponding to the map for the displacement quadratic $x_{1,i}^{(j)} x_{1,i}^{(k)}$. Similarly, multiplying the jth velocity map with the kth (as in Eq. (31)) and substituting for the drift coefficients $a^{(j)}(\bar{X}_{i-1}, t_{i-1})h$ and $a^{(k)}(\bar{X}_{i-1}, t_{i-1})h$, we get

$$\begin{aligned}
x_{2,i}^{(j)} x_{2,i}^{(k)} = {} & x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} + [C(\bar{X}_{i-1}, t_{i-1})^2 x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} + K(\bar{X}_{i-1}, t_{i-1})^2 x_{1,i-1}^{(j)} x_{1,i-1}^{(k)} + F^2(t) \\
& + C(\bar{X}_{i-1}, t_{i-1}) K(\bar{X}_{i-1}, t_{i-1}) [x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} + x_{1,i-1}^{(k)} x_{2,i-1}^{(j)}] \\
& - F(t)\{C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} + K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)}\} \\
& - F(t)\{C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} + K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)}\}] h^2 \\
& + \sum_{r,l=1}^{q} \sigma_r^{(j)} \sigma_l^{(k)} (\bar{X}_{i-1}, t_{i-1}) I_r I_l \\
& + [-2C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} \\
& - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)} x_{2,i-1}^{(j)}] h \\
& + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) [-C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)} + F(t)] h I_r \\
& + \sum_{l=1}^{q} \sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1}) [-C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)} + F(t)] h I_l \\
& + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} I_r + \sum_{l=1}^{q} \sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} I_l + R_{E,v^2}^{(j,k)} \quad (j,k) \in [1,n] \times [1,n].
\end{aligned} \tag{33}$$

Now multiplying the jth component of the displacement map with the kth one of the velocity map, we get the following system of cross-quadratic maps:

$$
\begin{aligned}
x_{1,i}^{(j)}x_{2,i}^{(k)} &= x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{2,i-1}^{(j)}x_{2,i-1}^{(k)}h + + \sum_{r=1}^{q}\sigma_r^{(k)}(\bar{X}_{i-1},t_{i-1})I_r[x_{1,i-1}^{(j)} + x_{2,i-1}^{(j)}h] \\
&+ [-C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(k)}x_{1,i-1}^{(j)}h - K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(j)}x_{1,i-1}^{(k)}h + F(t)x_{1,i-1}^{(j)}h] \\
&+ [-C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(j)}x_{2,i-1}^{(k)}h^2 - K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(j)}x_{2,i-1}^{(k)}h^2 + F(t)x_{2,i-1}^{(j)}h^2] + R_{E,dv}^{(j,k)} \\
(j,k) &\in [1,n]\times[1,n].
\end{aligned} \tag{34}
$$

We note that $R_{E,v^2}^{(j,k)}$ and $R_{E,dv}^{(j,k)}$ in Eqs. (33,34) are the (j,k)th components of the remainder vectors R_{E,v^2} and $R_{E,dv}$ corresponding to Euler maps for velocity quadratic and displacement–velocity cross-quadratic, respectively. It is clear that we can reduce variances associated with the second-order response processes (i.e., the left-hand side (LHS) of Eqs. (32)–(34)) provided that the random variables $I_r = N(0,h)$ and $I_r^2 = N(h,3h^2)$ are weakly modelled as two independent random variables λ_r and $\lambda_{r,sq}$ (respectively) with their means unchanged and the variance reduced (say, by a factor $k>1$). It is of interest to note that unlike the strong form of the identity $I_r^2 = (I_r)^2$, for the weakly modelled random, we have $\lambda_{r,sq} \neq (\lambda_r)^2$. Also, note that I_r and I_l are statistically independent for $r \neq l$ and hence $I_r I_l$ may be weakly modelled as being identically zero. Now, we know the stochastic Euler method to have local and global orders of accuracy of $O(h)$ and $O(\sqrt{h})$, respectively. We can therefore treat all random terms having a mean of $O(h^2)$ or more to be identically zero without affecting the order of accuracy of our variance reduction algorithm. With this in mind, while deriving the augmented system of SDEs for the weak, variance-reduced variables based on the stochastic Euler map, we require to construct SDEs for all such dependent variables which constitute a polynomial of degree 2 (i.e., variables of the forms $x_1^{(j)}, x_2^{(j)}$, and $x_1^{p_j,(j)}x_2^{q_k,(k)}$ such that $p_j + q_k = 2, j,k \in [1,n]$). This is because of the fact that SDEs for variables of the form $x_1^{p_j,(j)}x_2^{q_k,(k)}$ with $p_j + q_k = 2$ have stochastic terms containing I_r^2 whose mean is of $O(h)$. However, to construct such augmented equations based on a stochastic integrator of higher order than the Euler scheme (such as the weak stochastic Newmark scheme), we would need to find out SDEs for terms of the form $x_1^{p_j,(j)}x_2^{q_k,(k)}$ with $p_j + q_k > 2, \quad p_j, q_k \in Z^+$. Moreover, it is essential to frame a general set of rules to appropriately write down the drift and diffusion fields of the augmented system of SDEs. We denote variables of the form $x_1^{p_j,(j)}x_2^{q_k,(k)}$ to be unmixed (i.e., to have unmixed powers) if either $p_j = 0$ or $q_k = 0$. Otherwise, we call such a variable to be of the mixed type. Then, while forming the drift and diffusions terms for the variance-reduced augmented SDEs based on the Euler scheme, higher degree terms (higher than those present as dependent variables in the augmented equations) are decomposed into constituent terms in a way that the highest-degree unmixed terms of the type $x_1^{2,(j)}$ or $x_2^{2,(k)}$ are given the highest preference followed by mixed terms of the form $x_1^{(j)}x_2^{(j)}$ and the lowest degree terms, $x_1^{(j)}$ or $x_2^{(j)}$, in that order. Thus, for a term of the form $x_1^{p_j,(j)}x_2^{q_k,(k)}$ with $p_j + q_k > 2$, these rules may be stated in a more precise manner as follows:

$$p_j \to \text{even}; \quad q_k \to \text{even}; \quad x_1^{p_j,(j)}x_2^{q_k,(k)} = [x_1^{2,(j)}]^{p_j/2}[x_2^{2,(k)}]^{q_k/2}, \tag{35}$$

$$p_j \to \text{even}; \quad q_k \to \text{odd}; \quad x_1^{p_j,(j)}x_2^{q_k,(k)} = [x_1^{2,(j)}]^{p_j/2}[x_2^{2,(k)}]^{(q_k-1)/2}[x_2^{(k)}], \tag{36}$$

$$p_j \to \text{odd}; \quad q_k \to \text{even}; \quad x_1^{p_j,(j)}x_2^{q_k,(k)} = [x_1^{2,(j)}]^{(p_j-1)/2}[x_2^{2,(j)}]^{q_k/2}[x_1^{(j)}], \tag{37}$$

$$p_j \to \text{odd}; \quad q_k \to \text{odd}; \quad x_1^{p_j,(j)}x_2^{q_k,(k)} = [x_1^{2,(j)}]^{(p_j-1)/2}[x_2^{2,(k)}]^{(q_k-1)/2}[x_{1,i}^{(j)}x_{2,i}^{(k)}] \tag{38}$$

… and so on. Using the above decompositions, higher degree terms on the right-hand side (RHS) of maps (30)–(34) may be written only in terms of the set of dependent variables. This will enable us to close the system of maps and thus to explicitly solve for the dependent variables at $t = t_i$ based on their initial conditions at $t = t_{i-1}$. For example,

$$p_1 = 4, \quad q_k = 0; \quad x_1^{4,(j)} x_2^{0,(k)} = [x_1^{2,(j)}]^2,$$
$$p_1 = 5, \quad q_k = 5; \quad x_1^{5,(j)} x_2^{5,(k)} = [x_1^{2,(j)}]^2 [x_2^{2,(k)}]^2 [x_1^{(j)} x_2^{(k)}],$$
$$p_1 = 4, \quad q_k = 5; \quad x_1^{4,(j)} x_2^{5,(k)} = [x_1^{2,(j)}]^2 [x_2^{2,(k)}]^2 [x_2^{(k)}].$$

The presently adopted WIEM formulation is precisely similar to that of the WEM except that there is an implicitness parameter associated with coefficients of the highest powers of the step size, 'h', in each of the augmented system of maps. Thus, to begin with, the strong form of the equations gets modified to (provided that remainder terms are removed)

$$x_{1,i}^{(j)} = x_{1,i-1}^{(j)} + \alpha x_{2,i-1}^{(j)} h + (1-\alpha) x_{2,i}^{(j)} h + R_{IE,d}^{(j)} \quad j \in [1, n], \tag{39}$$

$$x_{2,i}^{(j)} = x_{2,i-1}^{(j)} + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) I_r + \beta a^{(j)}(\bar{X}_{i-1}, t_{i-1}) h + (1-\beta) a^{(j)}(\bar{X}_i, t_i) h + R_{IE,v}^{(j)} \quad j \in [1, n], \tag{40}$$

where α, β are implicitness parameters. Multiplying the jth displacement map with the kth (as in Eq. (30)) followed by the introduction of another implicitness parameter, χ, we get

$$x_{1,i}^{(j)} x_{1,i}^{(k)} = x_{1,i-1}^{(j)} x_{1,i-1}^{(k)} + x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} h + x_{1,i-1}^{(k)} x_{2,i-1}^{(j)} h + \chi x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} h^2 + (1-\chi) x_{2,i}^{(j)} x_{2,i}^{(k)} h^2 + R_{IE,d^2}^{(j,k)}$$
$$(j,k) \in [1,n] \times [1,n], \tag{41}$$

where $R_{IE,d^2}^{(j,k)}$ is the remainder term corresponding to the map for the displacement quadratic $x_{1,i}^{(j)} x_{1,i}^{(k)}$. Similarly, multiplying the jth velocity map with the kth (as in Eq. (31)) and substituting for the drift coefficients $a^{(j)}(\bar{X}_{i-1}, t_{i-1})h$ and $a^{(k)}(\bar{X}_{i-1}, t_{i-1})h$, with yet another implicitness parameter θ result in

$$\begin{aligned}
x_{2,i}^{(j)} x_{2,i}^{(k)} = {} & x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} + \theta[C(\bar{X}_{i-1}, t_{i-1})^2 x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} + K(\bar{X}_{i-1}, t_{i-1})^2 x_{1,i-1}^{(j)} x_{1,i-1}^{(k)} \\
& + F^2(t) + C(\bar{X}_{i-1}, t_{i-1}) K(\bar{X}_{i-1}, t_{i-1})[x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} + x_{1,i-1}^{(k)} x_{2,i-1}^{(j)}] \\
& - F(t)\{C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} + K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)}\} \\
& - F(t)\{C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} + K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)}\}] h^2 \\
& + (1-\theta)[C(\bar{X}_i, t_i)^2 x_{2,i}^{(j)} x_{2,i}^{(k)} + K(\bar{X}_i, t_i)^2 x_{1,i}^{(j)} x_{1,i}^{(k)} + F^2(t) \\
& + C(\bar{X}_i, t_i) K(\bar{X}_i, t_i)[x_{1,i}^{(j)} x_{2,i}^{(k)} + x_{1,i}^{(k)} x_{2,i}^{(j)}]. \\
& - F(t)\{C(\bar{X}_i, t_i) x_{2,i}^{(j)} + K(\bar{X}_i, t_i) x_{1,i}^{(j)}\} \\
& - F(t)\{C(\bar{X}_i, t_i) x_{2,i}^{(k)} + K(\bar{X}_i, t_i) x_{1,i}^{(k)}\}] h^2 \\
& + \sum_{r,l=1}^{q} \sigma_r^{(j)} \sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1}) I_r I_l \\
& + [-2C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} x_{2,i-1}^{(k)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)} x_{2,i-1}^{(k)} \\
& - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)} x_{2,i-1}^{(j)}] h. \\
& + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1})[-C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(k)} + F(t)] h I_r \\
& + \sum_{l=1}^{q} \sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1})[-C(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} - K(\bar{X}_{i-1}, t_{i-1}) x_{1,i-1}^{(j)} \\
& + F(t)] h I_l + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(k)} I_r + \sum_{l=1}^{q} \sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1}) x_{2,i-1}^{(j)} I_l + R_{IE,v^2}^{(j,k)} \quad (j,k) \in [1,n] \times [1,n].
\end{aligned} \tag{42}$$

Now multiplying the jth component of the displacement map with the kth one of the velocity map equation we get the following system of cross-quadratic maps:

$$\begin{aligned} x_{1,i}^{(j)}x_{2,i}^{(k)} &= x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{2,i-1}^{(j)}hx_{2,i-1}^{(k)}h + + \sum_{l=1}^{q}\sigma_l^{(k)}(\bar{X}_{i-1}, t_{i-1})I_l[x_{1,i-1}^{(j)} + x_{2,i-1}^{(j)}h] \\ &+ [-C(\bar{X}_{i-1}, t_{i-1})x_{2,i-1}^{(j)}x_{1,i-1}^{(k)}h - K(\bar{X}_{i-1}, t_{i-1})x_{1,i-1}^{(j)}x_{1,i-1}^{(k)}h + F(t)x_{1,i-1}^{(j)}h] \\ &+ \vartheta[-C(\bar{X}_{i-1}, t_{i-1})x_{2,i-1}^{(j)}x_{2,i-1}^{(k)}h^2 - K(\bar{X}_{i-1}, t_{i-1})x_{1,i-1}^{(j)}x_{2,i-1}^{(k)}h^2 + F(t)x_{2,i-1}^{(j)}h^2] \\ &+ (1-\vartheta)[-C(\bar{X}_i, t_i)x_{2,i}^{(j)}x_{2,i}^{(k)}h^2 - K(\bar{X}_i, t_i)x_{1,i}^{(j)}x_{2,i}^{(k)}h^2 + F(t)x_{2,i}^{(j)}h^2] + R_{IE,dv}^{(j,k)} \quad (j,k) \in [1,n]\times[1,n], \end{aligned} \tag{43}$$

where ϑ is last of the set of implicitness parameters introduced in forming the augmented maps. We note that $R_{IE,v^2}^{(j,k)}$ and $R_{IE,dv}^{(j,k)}$ in Eqs. (42,43) are the (j,k)th components of the remainder vectors R_{IE,v^2} and $R_{IE,dv}$ corresponding to implicit Euler maps for velocity quadratic and displacement–velocity cross-quadratic, respectively. Once the random variables (MSIs) I_r and I_rI_l are replaced by their variance-reduced weak counterparts (precisely the same as in WEM), we obtain the WIEM maps. As in the case of WEM, WIEM also has a local order $O(h)$ and thus we have modelled the augmented variables up to second powers only.

Now consider the SNM, the displacement and velocity maps are given by (removing the remainder terms)

$$\begin{aligned} x_{1,i}^{(j)} &= x_{1,i-1}^{(j)} + x_{2,i-1}^{(j)}h + \sum_{r=1}^{q}\sigma_r(\bar{X}_{i-1}, t_{i-1})I_{r0} \\ &+ 0.5\alpha a^{(j)}(\bar{X}_{i-1}, t_{i-1})h^2 + 0.5(1-\alpha)a^{(j)}(\bar{X}_i, t_i)h^2 + R_{\mathrm{SNM},d}^{(j)} \quad j \in [1,n], \end{aligned} \tag{44}$$

$$x_{2,i}^{(j)} = x_{2,i-1}^{(j)} + \sum_{r=1}^{q}\sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1})I_r + \beta a^{(j)}(\bar{X}_{i-1}, t_{i-1})h + (1-\beta)a^{(j)}(\bar{X}_i, t_i)h + R_{\mathrm{SNM},v}^{(j)} \quad j \in [1,n]. \tag{45}$$

As the SNM is derived by expanding the displacement and velocity components up to $O(h^2)$ and $O(h)$, respectively, for a given time-step size 'h' ([23,24]), we should be able to derive the variance-reduced WSNM by including in the displacement maps all stochastic terms with expectations of $O(h^2)$. Similarly for the velocity maps, we need only to account for stochastic terms with expectations of up to $O(h)$. Now we note that the displacement map (44) contains the MSIs $I_{r0}{\sim}N(0,h^3/3) r \in [1,q]$, whose squares are given by $I_{r0}^2{\sim}N(h^3/3, h^6/3)$. So it should suffice to model the displacement up to second powers as augmented variables. On the other hand, the velocity equation (45) has stochastic terms $I_r{\sim}N(0,h)$. Since, $I_r^2{\sim}N(h, 3h^2)$, $I_r^3{\sim}N(0, 15h^3)$ and $I_r^4{\sim}N(3h^2, 105h^4)$, it is enough to consider modelling of augmented variables corresponding up to only the second powers of velocity components. The statistical modelling of these random variables is derived in Appendix B. For the cross-terms (i.e., containing both velocity and displacement components) we need to model only those with mean of order up to $O(h^2)$. Finally, then, we need to model the following MSIs for the augmented system of SNM equations $I_{r0}{\sim}N(0,h^3/3)$, $I_{r0}^2{\sim}N(h^3/3, h^6/3)$, $I_rI_{r0}{\sim}N(0.5h^2, 5h^4/6)$, $I_r{\sim}N(0,h)$, $I_r^2{\sim}N(h, 3h^2) r \in [1,q]$. In other words, we need to write augmented maps for the following cross-terms: $x_1^{(j)}x_1^{(k)}, x_2^{(j)}x_2^{(k)}, x_1^{(j)}x_2^{(k)}$; $j,k \in [1,n]$. We provide further details of the augmented set of maps for the strong forms of SNM in Appendix A. In the process, we have made use of the five implicitness parameters, viz., $\alpha, \beta, \chi, \theta$ and ϑ. In their original form (i.e., without any imposed requirement of variance reduction), these maps would contain the following MSIs:

$$I_{r0}; I_{ro}^2; I_{r0}I_r; I_r \quad \text{and} \quad I_r^2. \tag{46}$$

We may thus obtain a variance-reduced version of the WSNM maps (i.e., the weak maps) by replacing the above MSIs by the following random variables (in the same order as they appear above) with k-fold reduced variances:

$$\begin{aligned}
&\eta_r \sim N(0, h^3/3k),\\
&\eta_{r,sq} \sim N(h^3/3, h^6/3k),\\
&\kappa_r \sim N(0.5h^2, 5h^4/6k),\\
&\lambda_r \sim N(0, h/k),\\
&\lambda_{r,sq}^{;} \sim N(h, 3h^2/k).
\end{aligned} \tag{47}$$

Also note that I_{r0} and I_{l0} are statistically independent for $r \neq l$ and hence $I_{r0}I_{l0}$ may be weakly modelled as being identically zero. Similar observations are valid for such random terms containing $I_{r0}I_l$ and I_rI_l. Indeed, there is no need to model the variance-reduced weak random variables (as listed in Eq. (47)) as Gaussian and any other probability distribution with the same mean and variance would do the job. A further simplification is then possible by exploiting the non-uniqueness in the probability distribution functions of the weakly modelled random variables. We provide a more detailed illustration of this point in Section 4. Now if we decompose any higher degree mixed term, say of the form, $x_1^{p_j,(j)} x_2^{q_k,(k)}$ such that, $p_j + q_k > 2$, then the constituent terms (factors) in the decomposed term should be formed using the following order of preference:

$$x_1^{2,(j)} \to x_1^{(j)} x_2^{(k)} \to x_2^{2,(j)} \to x_1^{(j)} \to x_2^{(j)}. \tag{48}$$

Indeed, for a given dynamical system, we may need to decompose quite a large number of higher degree terms. Only a few examples are cited below:

$$\begin{aligned}
&\text{Case 1}: \ q_k \to \text{even}; \quad \text{MOD}(q_k/2) = 0,\\
&\text{Sub-case (a)}: \ \text{MOD}(p_j/2) = 0 \Rightarrow x_1^{p_j,(j)} x_2^{q_k,(k)} = [x_1^{2,(j)}]^{p_j/2}[x_2^{2,(k)}]^{q_k/2},\\
&\text{Sub-case (b)}: \ \text{MOD}(p_j/2) = 1 \Rightarrow x_1^{p_j,(j)} x_2^{q_k,(k)} = [x_1^{2,(j)}]^{(p_j-1)/2}[x_2^{2,(k)}]^{q_k/2}[x_1^{(j)}].
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\text{Case 2}: \ q_k \to \text{even}; \ \text{MOD}(q_k/2) = 1,\\
&\text{Sub-case (a)}: \ \text{MOD}(p_j/2) = 0 \Rightarrow x_1^{p_j,(j)} x_2^{q_k,(k)} = [x_1^{2,(j)}]^{p_j/2}[x_2^{2,(k)}]^{q_k/2}[x_2^{(k)}],\\
&\text{Sub-case (b)}: \ \text{MOD}(p_j/2) = 1 \Rightarrow x_1^{p_j,(j)} x_2^{q_k,(k)} = [x_1^{2,(j)}]^{(p_j-1)/2}[x_2^{2,(k)}]^{q_k/2}[x_1^{(j)} x_2^{(k)}],
\end{aligned} \tag{50}$$

where $\text{MOD}(p/l)$ is a function that returns the remainder when p is divided by l.

3. Error estimates

We aim at deriving the weak error orders of (local) convergence of displacement and velocity components by the variance-reduced methods (i.e., WEM, WIEM and WSNM) as formulated above and thus finding a mathematical basis for the proposed procedures. However, the material reported on this section has nothing to do with the computer implementation of the numerical procedure, developed in the previous section and further illustrated with quite a few numerical examples in the following section.

Definition. A function $\psi(\bar{X}), \bar{X} \in \mathbf{R}^{2n}$, $Q \in R^+$ is said to belong to the class C_ρ, $\rho > 0$ when

$$||\psi(\bar{X})|| \leqslant Q(1 + ||\bar{X}||^\rho). \tag{51}$$

In what follows, it is assumed that $[M]$ is non-singular and that all the elements of the (possibly state dependent) damping and stiffness matrices, $C(X, \dot{X}, t), K(X, \dot{X}, t)$, as well as those of vectors $G_r(X, \dot{X}, t), r = 1, \ldots, q$, belong to C_ρ. It then follows that the drift and diffusion vectors, $\mathbf{a}(\bar{X}, t)$ and $\mathbf{b}_r(\bar{X}, t)$ also belong to C_ρ.

Proposition 1. *Under the above conditions (following Eq. (51)), Lipschitz boundedness (Eq. (4)), and the additional conditions that* $L^m C_{ij}(\bar{X}, t), L^m K_{ij}(\bar{X}, t) \in C_\rho$ (for $m = 1, 2, \ldots$), *we have the following inequalities for* *the remainder vectors* $R_{E,d} = \{R_d^{(j)}\}, R_{E,v}, R_{E,d^2} = \{R_{E,d^2}^{(i,j)}\}$, $R_{E,v^2}, R_{E,dv}$ (*Euler*), $R_{IE,d}, R_{IE,v}$, R_{IE,d^2}, R_{IE,v^2}, $R_{IE,dv}$ (*implicit Euler*) *and* $R_{SNM,d}$, $R_{SNM,v}, R_{SNM,d^2}$, $R_{SNM,v^2}, R_{SNM,dv}$ (*Newmark*):

$$\begin{aligned}&||\mathbf{E}R_{E,d}|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{E,d^2}|| \leqslant Q(\bar{X})h^2,\\ &||\mathbf{E}R_{IE,d}|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{IE,d^2}|| \leqslant Q(\bar{X})h^2,\\ &||\mathbf{E}R_{\mathrm{SNM},d}|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{\mathrm{SNM},d^2}|| \leqslant Q(\bar{X})h^3,\end{aligned} \tag{52}$$

$$||\mathbf{E}R_{E,d}.I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{IE,d}.I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{\mathrm{SNM},d}.I_r|| \leqslant Q(\bar{X})h^3, \tag{53}$$

$$||\mathbf{E}R_{E,d}.I_r^2|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{IE,d}.I_r^2|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{\mathrm{SNM},d}.I_r^2|| \leqslant Q(\bar{X})h^3, \tag{54}$$

$$||\mathbf{E}R_{E,d^2}.I_r^2|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{IE,d^2}.I_r^2|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{\mathrm{SNM},d^2}.I_r^2|| \leqslant Q(\bar{X})h^4, \tag{55}$$

$$||\mathbf{E}R_{\mathrm{SNM},d}.I_{r0}|| \leqslant Q(\bar{X})h^4, \quad ||\mathbf{E}R_{\mathrm{SNM},d^2}.I_{r0}|| = Q(\bar{X})h^4, \quad ||\mathbf{E}R_{\mathrm{SNM},d^2}.I_{r0}^2|| \leqslant Q(\bar{X})h^5, \tag{56}$$

$$\begin{aligned}&||\mathbf{E}R_{E,v}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{E,v^2}|| \leqslant Q(\bar{X})h,\\ &||\mathbf{E}R_{IE,v}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{IE,v^2}|| \leqslant Q(\bar{X})h,\\ &||\mathbf{E}R_{\mathrm{SNM},v}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{\mathrm{SNM},v^2}|| \leqslant Q(\bar{X})h,\end{aligned} \tag{57}$$

$$||\mathbf{E}R_{E,v}.I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{IE,v}.I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{\mathrm{SNM},v}.I_r|| \leqslant Q(\bar{X})h^2, \tag{58}$$

$$||\mathbf{E}R_{E,v^2}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{IE,v^2}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{\mathrm{SNM},v^2}|| \leqslant Q(\bar{X})h, \tag{59}$$

$$||\mathbf{E}R_{E,dv}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{IE,dv}|| \leqslant Q(\bar{X})h, \quad ||\mathbf{E}R_{\mathrm{SNM},dv}|| \leqslant Q(\bar{X})h, \tag{60}$$

$$||\mathbf{E}R_{E,dv}I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{IE,dv}I_r|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{\mathrm{SNM},dv}I_r|| \leqslant Q(\bar{X})h^2, \tag{61}$$

$$||\mathbf{E}R_{E,dv}I_r^2|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{IE,dv}I_r^2|| \leqslant Q(\bar{X})h^2, \quad ||\mathbf{E}R_{\mathrm{SNM},dv}I_r^2|| \leqslant Q(\bar{X})h^2, \tag{62}$$

$$||\mathbf{E}R_{\mathrm{SNM},dv}I_{r0}|| \leqslant Q(\bar{X})h^3, \quad ||\mathbf{E}R_{\mathrm{SNM},dv}I_{r0}^2|| \leqslant Q(\bar{X})h^4, \quad ||\mathbf{E}R_{\mathrm{SNM},dv}I_rI_{r0}|| \leqslant Q(\bar{X})h^3, \tag{63}$$

where $Q(\bar{X}) \in C_\rho$ is independent of h.

Proof. Proofs of the above statements remainders corresponding to EM, IEM and SNM are all similar; hence the proof for only a few of the inequalities involving the EM is provided here. From the expression of $R_{E,d}$ (Eq. (9)), it follows that

$$||\mathbf{E}R_{E,d}|| \leqslant ||0.5E[a^{(j)}(\bar{X}_{i-1}, t_{i-1})h^2] + E\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} La^{(j)}(\bar{X}(s_2), s_2)\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s], \tag{64}$$

Since LC_{ij} and $LK_{ij} \in C_p$ one can find an even number $2l$ and a positive real constant Q such that

$$||L\mathbf{a}(\bar{X}(s), s)|| \leqslant Q(1 + ||\bar{X}(s)||^{2l}). \tag{65}$$

Substituting (65) into (64) and noting that $\mathbf{E}||\bar{X}(s)||^{2l}$ is bounded by some quantity $Q(1 + ||\bar{X}_{i-1}||^{2l})$ ([16]), the first inequality of (52) follows immediately. Next, it is seen from Eq. (9) that the lowest order terms in $R_{E,d}$ contain integrals of the types $I_{r0} = \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s} \mathrm{d}W_r(s_1)\,\mathrm{d}s$, which is of orders $O(h^{3/2})$. Thus, we obtain the second inequality of (52) by noting that the (j,k)th component of R_{E,d^2} is given by

$$R_{E,d^2}^{(j,k)} = R_{E,d}^{(j)}X_{E,i}^{(k)} + R_{E,d}^{(k)}X_{E,i}^{(j)} + R_{E,d}^{(j)}R_{E,d}^{(k)} \tag{66}$$

followed by taking expectations of both sides of the above equation and finally taking the Euclidean norm. For proving the first inequalities of (53), (54) and (55), we use the Bunyakowski–Schwarz inequality. Thus, we can show that the norm of the expectation of the product of the lowest order terms of $R_{E,d}$ with I_r (which is $O(\sqrt{h})$) is of order $O(h^2)$. In a precisely similarly way, we can show all the other inequalities to be valid too.

Now we introduce the following notations for n-dimensional exact, strong (with the subscript 'S' omitted for convenience) and weak (appearing with the subscript 'W') response increment vectors by Euler scheme as

$$\begin{aligned}
&\Delta_d = \{\Delta_d^{(j)} | j = 1,\ldots,n\} = X_i - X_{i-1}, \quad \Delta_{E,d} = \{\Delta_{E,d}^{(j)}\} = X_{E,i} - X_{i-1},\\
&\Delta_{\mathrm{WE},d} = \{\Delta_{\mathrm{WE},d}^{(j)}\} = X_{\mathrm{WE},i} - X_{i-1},\\
&\Delta_v = \{\Delta_v^{(j)}\} = \dot{X}_i - \dot{X}_{i-1}, \quad \Delta_{E,v} = \{\Delta_{E,v}^{(j)}\} = \dot{X}_{E,i} - \dot{X}_{i-1},\\
&\Delta_{\mathrm{WE},v} = \{\Delta_{\mathrm{WE},v}^{(j)}\} = \dot{X}_{\mathrm{WE},i} - \dot{X}_{i-1}.
\end{aligned} \tag{67}$$

Moreover, we introduce the following additional vectors (required for setting up the augmented system of equations):

$$\begin{aligned}
&\Delta_{dv} = \{\Delta_{dv}^{(j,k)}\} = \{X_i^{(j)}\dot{X}_i^{(k)} - X_{i-1}^{(j)}\dot{X}_{i-1}^{(k)} \quad j,k \in [1,n]\},\\
&\Delta_{d^2} = \{\Delta_{d^2}^{(j,k)}\} = \{X_i^{(j)}X_i^{(k)} - X_{i-1}^{(j)}X_{i-1}^{(k)}\},\\
&\Delta_{v^2} = \{\Delta_{v^2}^{(j,k)}\} = \{\dot{X}_i^{(j)}\dot{X}_i^{(k)} - \dot{X}_{i-1}^{(j)}\dot{X}_{i-1}^{(k)}\},\\
&\Delta_{E,d^2} = \{\Delta_{E,d^2}^{(j,k)}\} = \{X_{E,i}^{(j)}X_{E,i}^{(k)} - X_{i-1}^{(j)}X_{i-1}^{(k)}\},\\
&\Delta_{\mathrm{WE},d^2} = \{\Delta_{\mathrm{WE},d^2}^{(j,k)}\} = \{X_{\mathrm{WE},i}^{(j)}X_{\mathrm{WE},i}^{(k)} - X_{i-1}^{(j)}X_{i-1}^{(k)}\},\\
&\Delta_{E,v^2} = \{\Delta_{E,v^2}^{(j,k)}\} = \{\dot{X}_{E,i}^{(j)}\dot{X}_{E,i}^{(k)} - \dot{X}_{i-1}^{(j)}\dot{X}_{i-1}^{(k)}\},\\
&\Delta_{\mathrm{WE},v^2} = \{\Delta_{\mathrm{WE},v^2}^{(j,k)}\} = \{\dot{X}_{\mathrm{WE},i}^{(j)}\dot{X}_{\mathrm{WE},i}^{(k)} - \dot{X}_{i-1}^{(j)}\dot{X}_{i-1}^{(k)}\},\\
&\Delta_{E,dv} = \{\Delta_{E,dv}^{(j,k)}\} = \{X_{E,i}^{(j)}\dot{X}_{E,i}^{(k)} - X_{i-1}^{(j)}\dot{X}_{i-1}^{(k)}\},\\
&\Delta_{\mathrm{WE},dv} = \{\Delta_{\mathrm{WE},dv}^{(j,k)}\} = \{X_{\mathrm{WE},i}^{(j)}\dot{X}_{\mathrm{WE},i}^{(k)} - X_{i-1}^{(j)}\dot{X}_{i-1}^{(k)}\}.
\end{aligned} \tag{68}$$

Precisely, similar notations would be used for IEM and WSNM with the subscript 'E' and 'WE' replaced by 'IE' and 'WIE' or 'SN' or 'WSN', respectively (as the case may be). Now one has the following relations between strong and exact increments via SEM:

$$\begin{aligned}
&\Delta_d = \Delta_{E,d} + R_d, \quad \Delta_v = \Delta_{E,v} + R_{E,v}, \quad \Delta_{d^2} = \Delta_{E,d^2} + R_{E,d^2},\\
&\Delta_{v^2} = \Delta_{E,v^2} + R_{E,v^2}, \quad \Delta_{dv} = \Delta_{E,dv} + R_{E,dv}.
\end{aligned} \tag{69}$$

We will also use similar notations with other methods (WEM, IEM, WIEM, SNM and WSNM). It may be noted that, to reduce the complexities in notations, we assume that all response increments (or approximate solutions) obtained by a certain method are through a strong version of the method, unless stated otherwise. For instance, $X_{E,i}^{(j)}$ is the strong solution $X^{(j)}(t_i)$of the jth displacement component by the SEM. The following proposition establishes the 'closeness' (in terms of orders of h) of the first few moments of various elements of the augmented system of exact and strong increment vectors.

Proposition 2. *For Lipschitz bounded drift and diffusion vectors* (*Eq.* (4)), *we have the following inequalities for the SEM:*

$$||\mathbf{E}(\Pi_{j=1}^{p}\Delta_d^{(i_j)} - \Pi_{j=1}^{p}\Delta_{E,d}^{(i_j)})|| \leqslant Q_E(\bar{X})h^2, \tag{70}$$

$$||\mathbf{E}(\Pi_{j=1}^{p}\Delta_v^{(i_j)} - \Pi_{j=1}^{p}\Delta_{E,v}^{(i_j)})|| \leqslant Q_E(\bar{X})h, \tag{71}$$

$$||\mathbf{E}(\Pi_{\substack{j,k=1\\ j+k\leqslant p}}^{p}\Delta_{d^2}^{(i_j,i_k)} - \Pi_{\substack{j,k=1\\ j+k\leqslant p}}^{p}\Delta_{E,d^2}^{(i_j,i_k)})|| \leqslant Q_E(\bar{X})h^2, \tag{72}$$

$$||\mathbf{E}(\Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{v^2} - \Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{E,v^2})|| \leqslant Q_E(\bar{X})h, \tag{73}$$

$$||\mathbf{E}(\Pi_{\substack{j,k=1,\dots,p\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{dv} - \Pi_{\substack{j,k=1,\dots,p\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{E,dv})|| \leqslant Q_E(\bar{X})h^2, \tag{74}$$

where $p = 1,2,\dots$ and $i_j, i_k \in [1,\dots,n], Q_E(\bar{X}) \in C_\rho$. Further, one has

$$\mathbf{E}\Pi^{m}_{j=1}||\Delta^{(i_j)}_{E,d}|| \leqslant Q_E(\bar{X})h,\ \mathbf{E}\Pi^{m}_{j=1}||\Delta^{(i_j)}_{E,v}|| \leqslant Q_E(\bar{X})h, \quad m = 1,2,\dots, \tag{75}$$

$$\mathbf{E}\Pi_{\substack{j,k=1,\dots,p\\ j+k\leqslant p}}||\Delta^{(i_j,i_k)}_{E,d^2}|| \leqslant Q_E(\bar{X})h^2. \tag{76}$$

For the stochastic Newmark method, we have

$$||\mathbf{E}(\Pi^{p}_{j=1}\Delta^{(i_j)}_{d} - \Pi^{p}_{j=1}\Delta^{(i_j)}_{\mathrm{SNM},d})|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^3; \quad p = 1,2,\dots, \tag{77}$$

$$||\mathbf{E}(\Pi^{p}_{j=1}\Delta^{(i_j)}_{v} - \Pi^{p}_{j=1}\Delta^{(i_j)}_{\mathrm{SNM},v})|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h; \quad p = 1,2,\dots, \tag{78}$$

$$||\mathbf{E}(\Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{d^2} - \Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{\mathrm{SNM},d^2})|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^3; \quad p = 1,2,\dots, \tag{79}$$

$$||\mathbf{E}(\Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{v^2} - \Pi^{p}_{\substack{j,k=1\\ j+k\leqslant p}} \Delta^{(i_j,i_k)}_{\mathrm{SNM},v^2})|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h; \quad p = 1,2,\dots, \tag{80}$$

$$||\mathbf{E}(\Pi_{\substack{j=1,\dots,p\\ k=1,\dots,r}} \Delta^{(i_j,i_k)}_{dv} - \Pi_{\substack{j=1,\dots,p\\ k=1,\dots,r}} \Delta^{(i_j,i_k)}_{\mathrm{SNM},dv})|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^3; \quad p+r = 2,\dots, \tag{81}$$

where $i_j, i_k \in [1,\dots,n], Q_{\mathrm{SNM}}(\bar{X}) \in C_\rho$. Further, one has

$$\mathbf{E}\Pi^{m}_{j=1}||\Delta^{(i_j)}_{\mathrm{SNM},d}|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^4;\ \mathbf{E}\Pi^{m}_{j=1}||\Delta^{(i_j)}_{\mathrm{SNM},v}|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^3; \quad m = 1,2,\dots, \tag{82}$$

$$\mathbf{E}\Pi_{\substack{j=1,\dots,p\\ k=1,\dots,r}}||\Delta^{(i_j,i_k)}_{\mathrm{SNM},dv}|| \leqslant Q_{\mathrm{SNM}}(\bar{X})h^3 \quad \text{for } p+r \geqslant 2. \tag{83}$$

Proof. From Eq. (69) and given that local initial conditions are treated as deterministic (for computing the local order of convergence), it follows that

$$||\mathbf{E}(\Delta^{(i_j)}_{d} - \Delta^{(i_j)}_{E,d})|| = ||\mathbf{E}R^{(i_j)}_{E,d}|| \leqslant Q_E(\bar{X})h^2 \quad \text{(via proposition 1)}.$$

To prove inequality (70), say, for $p = 2$ (i.e., for the second moments of displacement increments), it is first noted (via Eq. (69)) that

$$\mathbf{E}(\Delta^{(i_j)}_{d}\Delta^{(i_k)}_{d} - \Delta^{(i_j)}_{E,d}\Delta^{(i_k)}_{E,d}) = \mathbf{E}(R^{(i_j)}_{E,d}R^{(i_k)}_{E,d} - \Delta^{(i_k)}_{E,d}R^{(i_j)}_{E,d} - \Delta^{(i_j)}_{E,d}R^{(i_k)}_{E,d}). \tag{84}$$

Now, using inequalities (52) one readily has $\mathbf{E}(\Delta^{(i_j)}_{E,d}R^{(i_k)}_{E,d}) \leqslant O(h^3)$, (via Bunyakovsky–Schwarz inequality). Thus the RHS of Eq. (73) is less than or equal to $O(h^2)$ as claimed. Similarly, inequality (70) for $p = 3$ and higher may also be proved.

Inequality (71) for $p = 1$ is also readily proved, given that $||\mathbf{E}(\Delta^{(i_j)}_{v} - \Delta^{(i_j)}_{E,v})|| = ||\mathbf{E}R^{(i_j)}_{E,v}|| \leqslant Q_E(\bar{X})h$. For $p = 2$ one has

$$\mathbf{E}(\Delta^{(i_j)}_{v}\Delta^{(i_k)}_{v} - \Delta^{(i_j)}_{E,v}\Delta^{(i_k)}_{E,v}) = \mathbf{E}(R^{(i_j)}_{E,v}R^{(i_k)}_{E,v} - \Delta^{(i_k)}_{E,v}R^{(i_j)}_{E,v} - \Delta^{(i_j)}_{E,v}R^{(i_k)}_{E,v}). \tag{85}$$

From Eq. (57) one has $\mathbf{E}(R^{(i_j)}_{\mathrm{WE},v}R^{(i_k)}_{\mathrm{WE},v}) \leqslant O(h^2)$, and it is only to be shown that the other two terms on the RHS of (74) are also at least $O(h^2)$ It may be noted that inequality (72) for $p = 1$ corresponds to (71) for $p = 2$. The rest of the proof is precisely based on similar arguments.

Proposition 3. *Let the Lipschitz boundedness requirement of Eq.* (4) *holds. Moreover, suppose that the following relations between the MSIs and their weak counterparts hold* ($r = 1,2,\ldots,q$):

$$\mathbf{E}\lambda_r = \mathbf{E}I_r = 0, \quad \mathbf{E}\eta_r = \mathbf{E}I_{r0} = 0, \tag{86}$$

$$\begin{aligned} \mathbf{E}(\eta_u\eta_r) &= \mathbf{E}I_{u0}I_{r0} = \delta_{ur}\frac{h^3}{3}, \\ \mathbf{E}(\lambda_u\lambda_r) &= \mathbf{E}(I_uI_r) = \delta_{ur}h, \\ \mathbf{E}(\lambda_r\eta_u) &= \mathbf{E}(I_rI_{u0}) = \delta_{ru}\frac{h^2}{2}, \end{aligned} \tag{87}$$

$$\begin{aligned} \mathbf{E}(\lambda_i\lambda_r\eta_j) = \mathbf{E}(I_iI_rI_{j0}) = 0, \quad \mathbf{E}(\lambda_{r,sq}\eta_u) = \mathbf{E}(I_r^2I_{u0}) = 0, \\ \mathbf{E}(\lambda_r\eta_{u,sq}) = \mathbf{E}(I_rI_{u0}^2) = 0, \quad \mathbf{E}(\lambda_{r,sq}\lambda_r) = \mathbf{E}(I_r^3) = 0, \end{aligned} \tag{88}$$

$$\mathbf{E}(\lambda_{r,sq}^2) = \mathbf{E}(I_r^4) = 3h^2, \quad \mathbf{E}(\lambda_{r,sq}\lambda_{u,sq}) = \mathbf{E}(I_r^2I_u^2) = h^2(r \neq u), \tag{89}$$

$$\mathbf{E}(\lambda_r\lambda_u\lambda_v\lambda_w) = \mathbf{E}(I_rI_uI_vI_w) = 0 \quad \text{(for all other cases)}, \tag{90}$$

$$\mathbf{E}(\lambda_r\lambda_u\lambda_v\lambda_w\lambda_j) = \mathbf{E}(I_rI_uI_vI_wI_j) = 0. \tag{91}$$

Then one has the following inequalities relating the strong and weak increments of the augmented Euler method:

$$||\mathbf{E}\left(\prod_{j=1}^{p}\varDelta_{E,d}^{(i_j)} - \prod_{j=1}^{p}\varDelta_{\mathrm{WE},d}^{(i_j)}\right)|| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2; \quad p = 1,2,\ldots, \tag{92}$$

$$||\mathbf{E}\left(\prod_{j=1}^{p}\varDelta_{E,v}^{(i_j)} - \prod_{j=1}^{p}\varDelta_{\mathrm{WE},v}^{(i_j)}\right)|| \leqslant Q_{\mathrm{WE}}(\bar{X})h; \quad p = 1,2,\ldots, \tag{93}$$

$$||\mathbf{E}\left(\prod_{\substack{j=1,\ldots,p\\k=1,\ldots,r}}\varDelta_{E,dv}^{(i_j,i_k)} - \prod_{\substack{j=1,\ldots,p\\k=1,\ldots,r}}\varDelta_{\mathrm{WE},dv}^{(i_j,i_k)}\right)|| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2; \quad p+r = 2,\ldots, \tag{94}$$

$$\begin{aligned} ||\mathbf{E}\left(\prod_{\substack{j,k=1\\j+k\leqslant p}}\varDelta_{E,d^2}^{(i_j,i_k)} - \prod_{\substack{j,k=1\\j+k\leqslant p}}\varDelta_{\mathrm{WE},d^2}^{(i_j,i_k)}\right)|| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2, \\ ||\mathbf{E}\left(\prod_{\substack{j,k=1\\j+k\leqslant p}}\varDelta_{E,v^2}^{(i_j,i_k)} - \prod_{\substack{j,k=1\\j+k\leqslant p}}\varDelta_{\mathrm{WE},v^2}^{(i_j,i_k)}\right)|| \leqslant Q_{\mathrm{WE}}(\bar{X})h, \end{aligned} \tag{95}$$

where $p = 1,2,\ldots$ and $i_j, i_k \in [1,\ldots,n]$, $Q_{\mathrm{WE}}(\bar{X}) \in C_\rho$. Further one has the following bounds on the weak increments corresponding to the augmented Euler map:

$$\begin{aligned} \mathbf{E}\prod_{j=1}^{m}\left\|\varDelta_{\mathrm{WE},d}^{(i_j)}\right\| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2, \quad \mathbf{E}\prod_{j=1}^{m}\left\|\varDelta_{\mathrm{WE},v}^{(i_j)}\right\| \leqslant Q_{\mathrm{WE}}(\bar{X})h; \quad m = 1,2,\ldots, \\ \mathbf{E}\prod_{\substack{j,k=1,\ldots,p\\j+k\leqslant p}}\left\|\varDelta_{\mathrm{WE},d^2}^{(i_j,i_k)}\right\| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2, \quad \mathbf{E}\prod_{\substack{j=1,\ldots,p\\k=1,\ldots,r}}\left\|\varDelta_{\mathrm{WE},dv}^{(i_j,i_k)}\right\| \leqslant Q_{\mathrm{WE}}(\bar{X})h^2. \end{aligned} \tag{96}$$

A similar set of inequalities for strong and weak increments, obtainable, respectively, from strong and weak forms of the augmented stochastic Newmark maps, are

$$\left\| \mathbf{E}\left(\prod_{j=1}^{p} \Delta_{\mathrm{SNM},d}^{(i_j)} - \prod_{j=1}^{p} \Delta_{\mathrm{WSNM},d}^{(i_j)} \right) \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^3; \quad p = 1, 2, \ldots, \tag{97}$$

$$\left\| \mathbf{E}\left(\prod_{j=1}^{p} \Delta_{\mathrm{SNM},v}^{(i_j)} - \prod_{j=1}^{p} \Delta_{\mathrm{WSNM},v}^{(i_j)} \right) \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h; \quad p = 1, 2, \ldots, \tag{98}$$

$$\left\| \mathbf{E}\left(\prod_{\substack{j,k=1 \\ j+k\leqslant p}}^{p} \Delta_{\mathrm{SNM},d^2}^{(i_j,i_k)} - \prod_{\substack{j,k=1 \\ j+k\leq p}}^{p} \Delta_{\mathrm{WSNM},d^2}^{(i_j,i_k)} \right) \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^3, \tag{99}$$

$$\left\| \mathbf{E}\left(\prod_{\substack{j,k=1 \\ j+k\leqslant p}}^{p} \Delta_{\mathrm{SNM},v^2}^{(i_j,i_k)} - \prod_{\substack{j,k=1 \\ j+k\leqslant p}}^{p} \Delta_{\mathrm{WSNM},v^2}^{(i_j,i_k)} \right) \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h, \tag{100}$$

$$\left\| \mathbf{E}\left(\prod_{\substack{j=1,\ldots,p \\ k=1,\ldots,r}} \Delta_{\mathrm{SNM},dv}^{(i_j,i_k)} - \prod_{\substack{j=1,\ldots,p \\ k=1,\ldots,r}} \Delta_{\mathrm{WSNM},dv}^{(i_j,i_k)} \right) \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^3; \quad p + r = 2, \ldots, \tag{101}$$

where $i_j, i_k \in [1, \ldots, n]$, $Q_{\mathrm{WSNM}}(\bar{X}) \in C_\rho$. Further, one has

$$\mathbf{E} \prod_{j=1}^{m} \left\| \Delta_{\mathrm{WSNM},d}^{(i_j)} \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^4, \quad \mathbf{E} \prod_{j=1}^{m} \left\| \Delta_{\mathrm{WSNM},v}^{(i_j)} \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^3; \quad m = 1, 2, \ldots, \tag{102}$$

$$\mathbf{E} \prod_{\substack{j=1,\ldots,p \\ k=1,\ldots,r}} \left\| \Delta_{\mathrm{WSNM},dv}^{(i_j,i_k)} \right\| \leqslant Q_{\mathrm{WSNM}}(\bar{X}) h^3 \quad \text{for } p + r \geqslant 2. \tag{103}$$

Proof. If all the inequalities of the above proposition were true then, together with inequalities of Proposition 2, this would have meant that at least all the moments of pathwise and weak increments of augmented state variables were of the same error order in h. Now, to begin with, we prove identities (86)–(91). Proof of the RHS of (86) directly follows from the definition of Wiener increments. For the RHS of (87), first note that $\mathbf{E} I_{u0} I_{r0} = 0$ because of independence of $W_u(t)$ and $W_r(t)$ for $r \neq u$. For $r = u$ one has the squared MSI $I_{r0}^2 = \{\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathrm{d}W_r(s)\mathrm{d}t\}^2$. To obtain its statistics, consider the linear SDEs $\mathrm{d}x = \mathrm{d}W_r$; $\mathrm{d}y = x\mathrm{d}t$ subjected to initial conditions $x(t_{i-1}) = y(t_{i-1}) = 0$. Then, using Ito's formula, $\mathrm{d}(I_{r0}^2) = \mathrm{d}(y^2) = 2xy\mathrm{d}t$ so that $\mathrm{d}\mathbf{E}(I_{r0}^2) = 2\mathbf{E}(xy)\mathrm{d}t$. Applying Ito's formula now to xy yields $\mathrm{d}(xy) = y\mathrm{d}W_r(t) + x^2\mathrm{d}t$ and hence one immediately arrives at $\mathbf{E}(xy) = \int_{t_{i-1}}^{t_i} \mathbf{E}(x^2)\mathrm{d}t = \int_{t_{i-1}}^{t_i} \mathbf{E}(W_r^2)\mathrm{d}t = h^2/2$. Finally, one gets $\mathbf{E}(I_{r0}^2) = 2\int_{t_{i-1}}^{t_i} \mathbf{E}(xy)\mathrm{d}t = h^3/3$. Next one can show that $\mathbf{E}(I_{r0} I_{uv0}) = 0$ using the fact that for all positive values of r, u, and v, at least one Wiener increment (either $\mathrm{d}W_r$ or $\mathrm{d}W_u$ or $\mathrm{d}W_v$) appears an odd number of times in the expression $I_{r0} I_{uv0}$. As one more instance, consider $\mathbf{E}(I_{ur}^2)$ and construct the linear SDEs: $\mathrm{d}x = \mathrm{d}W_u(t)$; $\mathrm{d}y = x\mathrm{d}W_r(t)$ (again subjected to zero initial conditions). Then, appealing once more to Ito's formula, $\mathrm{d}(I_{ur}^2) = \mathrm{d}(y^2) = 2xy\mathrm{d}W_r + x^2\delta_{ru}\mathrm{d}t$. Taking expectations of both the sides, taking into account the zero initial conditions, and noting that $\mathbf{E}(x^2(t)) = \mathbf{E}(W_u^2(t)) = t$, the required identity, i.e., $\mathbf{E}(I_{ur}^2) = 0.5h^2\delta_{ru}$ follows. In fact, all other identities corresponding to the right parts of the system of Eq. (88) may be proved (either using Ito's formula or via oddness considerations). For instance, consider the quantity $\mathrm{E}(I_i I_r I_{j0})$ and introduce the new Wiener processes

$B_k(t) = -W_k(t)$ for $k = 1, 2, \ldots, q$. Then one gets

$$E(I_i I_r I_{j0}) = \int_{t_{i-1}}^{t_i} \mathrm{d}W_i \int_{t_{i-1}}^{t_i} \mathrm{d}W_r \int_{t_{i-1}}^{s} \mathrm{d}W_j(s_1)\mathrm{d}s = -\int_{t_{i-1}}^{t_i} \mathrm{d}B_i \int_{t_{i-1}}^{t_i} \mathrm{d}B_r \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathrm{d}B_j(s_1)\,\mathrm{d}s = -\mathrm{E}(I_i I_r I_{j0}).$$

In this way, RHSs of (88)–(91) follow. Inequalities (92)–(103) may also be proved readily by explicitly writing down weak or strong expressions for the increments followed by taking norms of their expectations while using identities (86)–(91). Appendix B contains the details of derivations of the statistical properties of the MSIs.

Based on the preceding discussion, the main result on the variance-reduced augmented methods may be proposed as

Theorem 1. *Let all the conditions as stated in Propositions* 1–3 *hold. Also suppose that* $f_1(X, t) \in C_\rho$ *is a function of displacement alone and* $f_2(X, \dot{X}, t) \in C_\rho$ *is a function of both displacement and velocity vectors. Moreover, assume that all partial derivatives of* f_1 *with respect to* X *(up to an adequately high order, explained further in the proof) and those of* f_2 *with respect to* X *and* $\dot{X}$ *exist and they belong to the class* C_ρ *as well. Then one has, for the weak approximations* $\bar{X}_W = \{X_W, \dot{X}_W\}$, *the following inequalities*:

$$\|\mathbf{E}f_1(X) - \mathbf{E}f_1(X_W)\| \leqslant Q_d(\bar{X})h^a; \quad a = 2 \text{ for WEM, WIEM}; \quad a = 3 \text{ for WSNM}, \tag{104}$$

$$\|\mathbf{E}f_2(\bar{X}) - \mathbf{E}f_2(\bar{X}_W)\| \leqslant Q_{dv}(\bar{X})h^b; \quad b = 2 \text{ for WEM, WIEM and WSNM}. \tag{105}$$

Proof. We have already observed that

$$\mathbf{E}\prod_{j=1}^{m} \|\Delta_d^{(i_j)}\| \leqslant Q_d(\bar{X})h^a \quad \text{and} \quad \mathbf{E}\Pi_{\substack{j=0,\ldots,p\\ i=0,\ldots,r}} \|\Delta_d^{(i_j)} \Delta_v^{(i_k)}\| \leqslant Q_{dv}(\bar{X})h^b.$$

Now, we may proceed as follows for the proof of inequality (104). Expand $f_1(X_{i+1}) = f_1(X_i + \Delta_d)$ via a Taylor expansion in powers of the displacement increment vector $\Delta_d = X_{i+1} - X_i$ based at X_i:

$$\begin{aligned} f_1(X_{i+1}) &= f_1(X_i + \Delta_d) = f_1(X_i) + \sum_j f_{1,X^{(j)}}(X_i)\Delta_d^{(j)} + \frac{1}{2!}\sum_{j,k} f_{1,X^{(j)}X^{(k)}}(X_i)\Delta_{d^2}^{(j,k)} \\ &\quad + \frac{1}{3!}\sum_{j,k,l} f_{1,X^{(j)}X^{(k)}X^{(l)}}(X_i)\Delta_{d^2}^{(j,k)}\Delta_d^{(l)} + \cdots, \end{aligned} \tag{106}$$

where

$$f_{1,X^{(j)}}(X_i) \triangleq \left.\frac{\partial f_1(X)}{\partial X^{(j)}}\right|_{X=X_i}, \quad f_{1,X^{(j)}X^{(k)}}(X_i) \triangleq \left.\frac{\partial^2 f_1(X)}{\partial X^{(j)}\partial X^{(k)}}\right|_{X=X_i}$$

and so on. Upon the substitution of the strong Euler displacement increments into the above equation, we get

$$\begin{aligned} f_1(X_{i+1}) &= f_1\left(X_i + \left(\Delta_{E,d}^{(j)} + R_{E,d}^{(j)}\right)\right) = f_1(X_i) + \sum_j f_{1,X^{(j)}}(X_i)\left(\Delta_{E,d}^{(j)} + R_{E,d}^{(j)}\right) \\ &\quad + \frac{1}{2!}\sum_{j,k} f_{1,X^{(j)}X^{(k)}}(X_i)\left(\Delta_{E,d^2}^{(j,k)} + R_{E,d^2}^{(j,k)}\right) \\ &\quad + \frac{1}{3!}\sum_{j,k,l} f_{1,X^{(j)}X^{(k)}X(l)}(X_i)\left(\Delta_{E,d^2}^{(j,k)} + R_{E,d^2}^{(j,k)}\right)\left(\Delta_{E,d}^{(j)} + R_{E,d}^{(j)}\right) + \ldots. \end{aligned} \tag{107}$$

Taking expectations on both sides leads to

$$\begin{aligned} \mathbf{E}\left[f_1(X_{i+1})\right] &= f_1(X_i) + \sum_j f_{1,X^{(j)}}(X_i)\mathbf{E}\left[\Delta_{E,d}^{(j)}\right] + \frac{1}{2!}\sum_{j,k} f_{1,X^{(j)}X^{(k)}}(X_i)\mathbf{E}\left[\Delta_{E,d^2}^{(j,k)}\right] \\ &\quad + \frac{1}{3!}\sum_{j,k,l} f_{1,X^{(j)}X^{(k)}X(l)}(X_i)\mathbf{E}\left[\Delta_{E,d^2}^{(j,k)}\Delta_{E,d}^{(j)}\right] + \mathbf{E}\left[R_{f_1}\right], \end{aligned} \tag{108}$$

where

$$\mathbf{E}\left[R_{f_1}\right] = \mathbf{E}\left[\sum_j R^{(j)}_{E,d} + \sum_{j,k} R^{(j,k)}_{E,d^2} + \sum_{j,k,l}\left\{R^{(j,k)}_{E,d^2}R^{(l)}_{E,d} + R^{(j,k)}_{E,d^2}\Delta^{(l)}_{E,d} + R^{(j)}_{E,d}\Delta^{(k,l)}_{E,d^2}\right\}\right]. \tag{109}$$

Now, using the triangle inequality

$$\mathbf{E}||\Delta_{E,d}|| = \mathbf{E}||(\Delta_{E,d} - \Delta_{\mathrm{WE},d}) + \Delta_{\mathrm{WE},d}|| \leqslant \mathbf{E}||(\Delta_{E,d} - \Delta_{\mathrm{WE},d})|| + \mathbf{E}||\Delta_{\mathrm{WE},d}||. \tag{110}$$

By Proposition 2, we know that

$$\mathbf{E}||\Delta_{E,d}|| \leqslant \mathbf{E}||(\Delta_{E,d} - \Delta_{\mathrm{WE},d})|| + \mathbf{E}||\Delta_{\mathrm{WE},d}|| \leqslant O(h^2) + \mathbf{E}||\Delta_{\mathrm{WE},d}||. \tag{111}$$

Similar inequalities between any other strong and weak increments or their powers may also be written in a likewise manner. Substituting them into the RHS of Eq. (107), we readily arrive at the first of the proposed inequalities (104). The steps to prove inequality (105) remain precisely the same. In addition, the procedure to prove similar inequalities for WIEM and WSNM also remain the same.

4. Illustrative examples

4.1. The Duffing equation under only additive noise

In order to keep the numerical illustration simple and focussed, consider the nonlinear second-order SDE corresponding to a singe-degree-of-freedom (sdof) hardening Duffing oscillator under an additive white noise, described by the following equation:

$$\ddot{x} + C\dot{x} + K_1x + K_2x^3 = \sigma\dot{W}(t). \tag{112}$$

As the differentiation of the Weiner process $W(t)$ cannot be mathematically accomplished in a pathwise sense, the above equation is written in the following incremental state space form:

$$\begin{aligned}\mathrm{d}x(t) &= \dot{x}(t)\,\mathrm{d}t,\\ \mathrm{d}\dot{x}(t) &= a(x,\dot{x},t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_1(t),\end{aligned} \tag{113}$$

where the velocity drift coefficient is given by $a(x,\dot{x},t) = -C\dot{x} - K_1x - K_2x^3$. As we can verify from the formulations of Section 2, a large number of terms drop out in the augmented stochastic maps for response updates if the deterministic force $[F(t)]$ is absent and the noise is purely additive (i.e.,$\sigma^{(j)}_r = \sigma^{(j)}_r(\bar{X}_{i-1}, t_{i-1})$). Hence, all the examples considered hereunder satisfy the above two conditions.

In Eqs. (30)–(34) with $n = 1$, we get the Euler augmented set of variables $x = \{x_1, x_2, x_1^2, x_2^2, x_1x_2\}$:

$$x_{1,i} = [x_{1,i-1}] + [x_{2,i-1}]h, \tag{114}$$

$$x_{2,i} = [x_{2,i-1}] + \sigma\lambda_1 + [-C[x^2_{2,i-1}] - K_1[x_{1,i-1}x_{2,i-1}] - K_2[x_{1,i-1}x_{2,i-1}][x^2_{1,i-1}]]h, \tag{115}$$

$$x^2_{1,i} = [x^2_{1,i-1}] + [x^2_{2,i-1}]h^2 + 2[x_{1,i-1}x_{2,i-1}]h, \tag{116}$$

$$\begin{aligned}x^2_{2,i} = x^2_{2,i-1} &+ [C^2x^2_{2,i-1} + K_1^2[x^{2,(j)}_{1,i-1}] + K_2^2[x^{2,(j)}_{1,i-1}]^3 + 2CK_1x_{1,i-1}x_{2,i-1}\\ &+ 2K_1K_2[x_{1,i-1}]^2 + 2CK_2[x^2_{1,i-1}][x_{1,i-1}x_{2,i-1}]]h^2 + \sigma^2\lambda_{1,sq}\\ &+ 2[-C[x^2_{2,i-1}] - K_1[x_{1,i-1}x_{2,i-1}] - K_2[x_{1,i-1}x_{2,i-1}][x^2_{1,i-1}]]h\\ &+ 2\sigma[-Cx_{2,i-1} - K_1x_{1,i-1} - K_2[x_{1,i-1}x_{2,i-1}][x^2_{1,i-1}]]hI_1 + 2\sigma x_{2,i-1}\lambda_1,\end{aligned} \tag{117}$$

$$\begin{aligned}x_{1,i}x_{2,i} = x_{1,i-1}x_{2,i-1} &+ x^2_{2,i-1}h + \sigma\lambda_1[[x_{1,i-1}] + [x_{2,i-1}]h] + [-C[x_{2,i-1}x_{1,i-1}]h - K_1[x^2_{1,i-1}]h\\ &- K_2[x^2_{1,i-1}]^2h] + [-C[x^2_{2,i-1}]h^2 - K[x_{1,i-1}x_{2,i-1}]h^2 - K_2[x^2_{1,i-1}]^2[x_{1,i-1}x_{2,i-1}]h^2].\end{aligned} \tag{118}$$

Now, we generate the weak random variables, representing the MSIs, according to the following distribution:

$$\text{WEM}: P(\lambda_1 = \pm\sqrt{h/k}) = 0.50, \quad P(\lambda_{1,sq} = h \pm \sqrt{3h^2/k}) = 0.50. \tag{119}$$

We may derive similar expressions for WIEM by introducing the implicitness parameters using maps (39)–(43). In WIEM, the weak random variables that are required for generation, to represent the MSIs, are λ_1 and $\lambda_{1,sq}$ and their probability distribution is identically same as that given by WEM corresponding to Eq. (119). WSNM maps can also be derived in a likewise manner. The random variables associated with the maps representing the MSIs can be modelled using the following distributions:

$$\begin{aligned}
&\lambda_r \sim N(0, h/k),\\
&\eta_r \sim N(0, h^3/3k),\\
&\kappa_r \sim N(0.5h^2, 5h^4/6k),\\
&\eta_{r,sq} \sim N(h^3/3, h^6/3k),\\
&\lambda_{r,sq} \sim N(h, 3h^2/k).
\end{aligned} \tag{120}$$

However, as modelling of Gaussian random variables requires the usage of transcendental functions as sine, cosine and logarithm, the above distribution has been replaced by another probability distribution requiring less computational effort:

$$\begin{aligned}
&P(\lambda_r = \pm\sqrt{h/k}) = 0.50,\\
&\eta_r = 0.5h\lambda_r + \gamma_r h^{1.5}, \quad P(\gamma_r = \pm\sqrt{1/(12k)}) = 0.50,\\
&P(\lambda_{r,sq} = h \pm \sqrt{2h^2/k}) = 0.50,\\
&\kappa_r = (h/2)\lambda_{r,sq} + \tau_r h^2, \quad P(\tau_r = \pm\sqrt{1/(12k)}) = 0.50,\\
&\eta_{r,sq} = -(2/9)h^2\lambda_{r,sq} + h\kappa_r + h^3\varsigma_r, \quad P(\varsigma_r = 1/18 \pm \sqrt{5/(324k)}) = 0.50.
\end{aligned} \tag{121}$$

4.1.1. Results through the variance-reduced WEM

We have observed that expectations of the first few powers of the displacement and velocity components, simulated through Variance-reduced WEM (VR-WEM) with a very small ensemble size and through the classical form of the SEM with a much larger ensemble size, match very well for different values of the variance reduction parameter $k \gg 1$. However, to bring into focus the reduction in variances of computed expectations through VR-WEM vis-à-vis direct SEM for the same ensemble sizes, Figs. 1–3 plot time-histories of expectations of (some of) the first three powers of displacement and velocity components with a uniformly chosen ensemble size of 500. For obtaining these figures, we have chosen the system parameters as $C = 2.0$, $K_1 = 100$, $K_2 = 10$. The additive noise intensity parameter is $\sigma = 5.0$ and the step size is uniform at $h = 0.001$. While a visual inspection of Figs. 1–3 clearly shows a drastic reduction in variances of the expectation histories via VR-WEM, this is more explicitly brought out in Fig. 4, which shows the precise reduction in the variance of expectation histories corresponding to (some of the) second and third powers of displacement/velocity components for different choices of k. As anticipated, we can readily verify from these figures that the ratio of variance reduction (i.e., the ratio of short-time-averages of the variance histories via SEM and WEM) is reasonably close to the variance reduction parameter k. This is further brought out in Figs. 5(a) and (b), wherein we plot the histories of the variance reduction ratios for the first and second moments of the displacement. We may also note that the ratio of variance reductions corresponding to the displacement quadratic (as in Fig. 5b) is close to k^2. However, one must keep in mind that such observations are valid for oscillators (possibly nonlinear) driven by additive noises only, and not necessarily for oscillators with multiplicative noises (as there could be nonlinear interactions between k and the response components in the latter case).

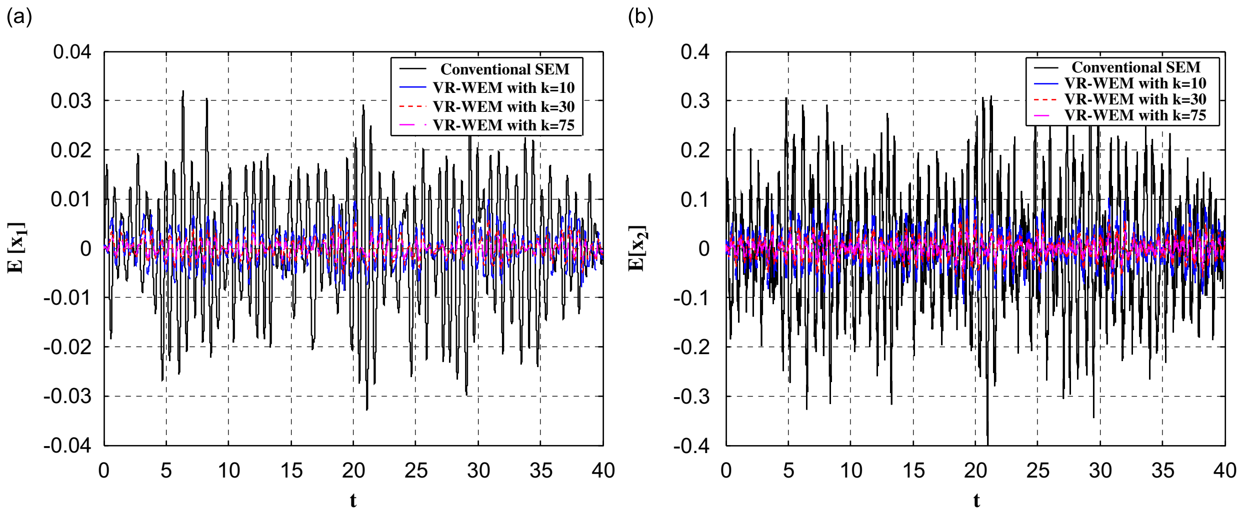

Fig. 1. First moment histories of Duffing equation under additive noise: (a) mean of displacement of (b) mean of velocity; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500.

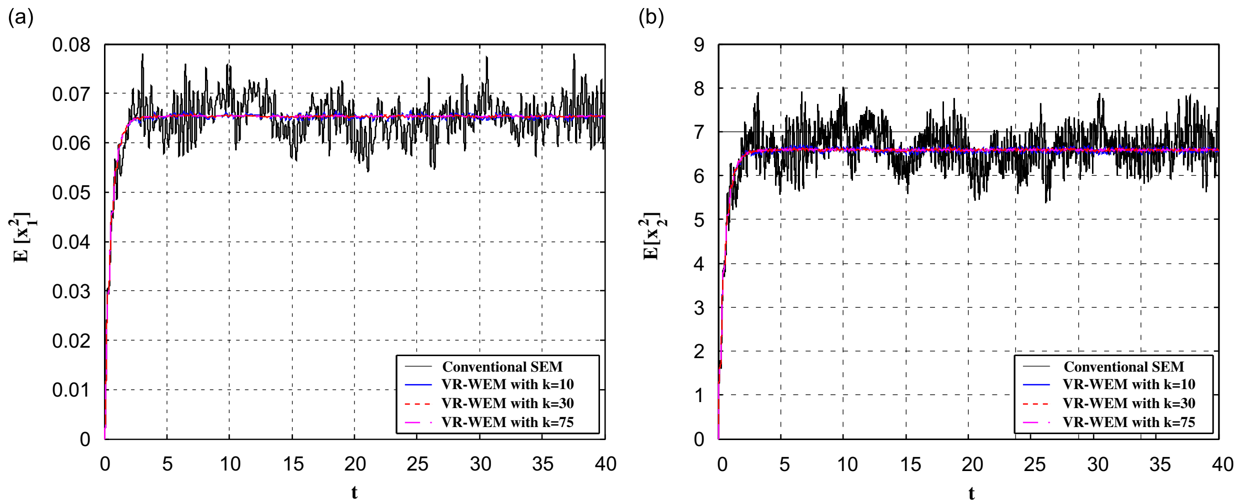

Fig. 2. Second moment histories of Duffing equation under additive noise: (a) expectation of displacement, (b) expectation of velocity quadratic; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500.

4.1.2. *Results through the Variance-reduced WIEM*

We have found enough numerical evidence to suggest that the Variance-reduced WIEM (VR-WIEM) generally leads to a smaller variance (of computed expectations) than the Variance-reduced VR-WEM when all other factors remain unchanged. To demonstrate this, we plot histories of the signed differences of variances of $\mathbf{E}[X_1^2]$ in Fig. 6 through WIEM and WEM for $k = 10$. It is evident that the signed differences of variances via WEM and WIEM stay much longer above zero, thereby confirming the last observation. Being an implicit method, WIEM also has a higher stochastic numerical stability [14] and numerical results through WIEM show a close correspondence with SEM (provided that the ensemble size for the latter scheme is large enough). As a few representative cases, we plot the histories of second and third powers of displacement/velocity components in Figs. 7(a) and 7(b) along with comparisons through the conventional SEM. As in the case of VR-WEM, simulations through VR-WIEM also lead to an approximately k-fold reduction in variance (again in the sense of short time averages) and this is substantiated in Fig. 8. The implementation of

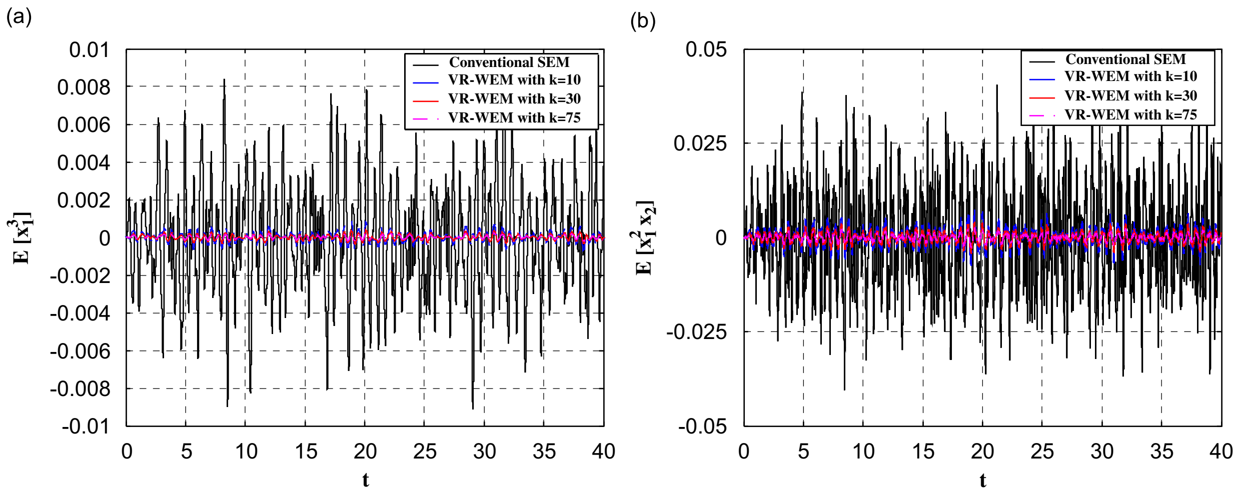

Fig. 3. Third moment histories of Duffing equation under additive noise: (a) expectation of displacement cube, (b) expectation of cross-moment of displacement-square and velocity; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 50.

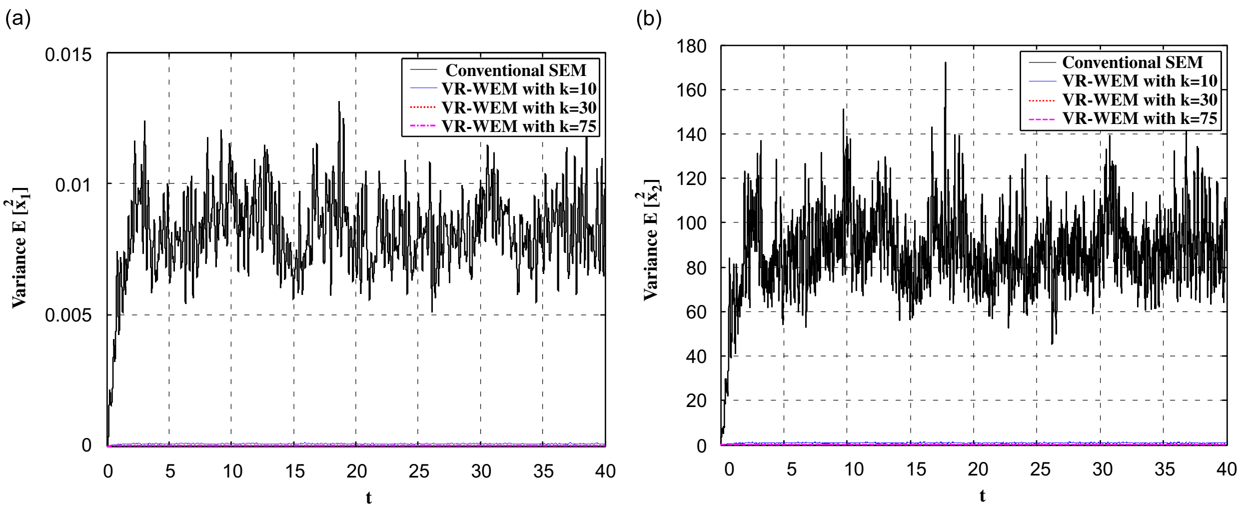

Fig. 4. Variance histories of Duffing equation under additive noise: (a) variance of displacement quadratic, (b) variance of velocity quadratic; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500.

VR-WIEM has so far used a uniform value of 0.50 for all the five implicitness parameters. Fig. 9 shows a couple of second moment history plots for several choices of these parameters. It is amply clear that the results are presently quite insensitive to such variations. Finally, it is noted that the ensemble size has consistently been fixed at 500. However, substantial variance reduction can also be generally achieved with far less number of samples.

4.1.3. *Results through VR-WSNM*

Variance-reduced histories of second and higher order moments through VR-WSNM are plotted, respectively, in Figs. 10 and 11, which also show comparisons with those via the conventional SNM in its weak form (WSNM) (i.e., with $k = 1$). All the previous observations regarding the k-fold reduction of variances are applicable in this case too and hence we do not discuss these issues further.

One specific reason that we have so far restricted our attention to a hardening Duffing oscillator under additive white noise is that the stationary probability density function (SPDF) of such a system may be exactly

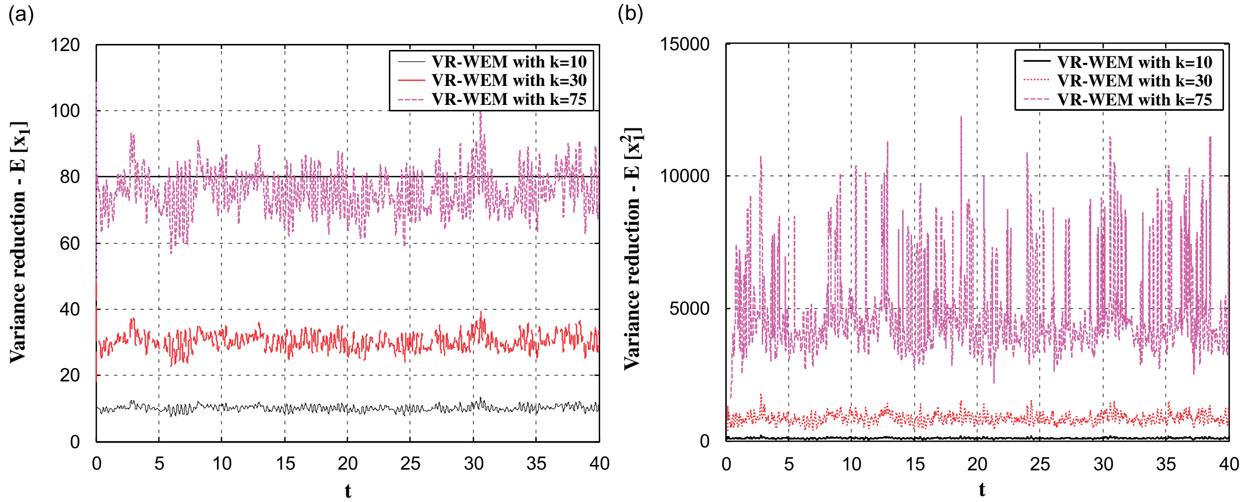

Fig. 5. Variance reduction-ratio histories of Duffing equation under additive noise: (a) ratio for displacement, (b) ratio for displacement quadratic; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500.

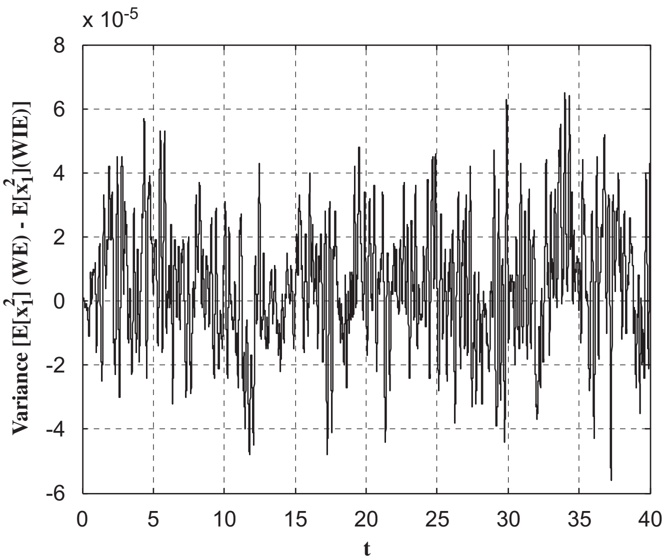

Fig. 6. Histories of difference of WEM-WIEM of Duffing equation under additive noise of displacement-quadratic ; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500, all implicit parameters $= 0.50$.

obtained via a closed form, exact solution of the reduced Fokker–Planck equation. The stationary density is thus given by (see Ref. [9])

$$p(x, \dot{x}) = A \exp\left(-\frac{2}{\sigma^2} C\left(\frac{\dot{x}^2}{2} + K_1 \frac{x^2}{2} + K_2 \frac{x^4}{4}\right)\right) \tag{122}$$

with A being a normalization constant to ensure that the infinite double integral of the SPDF is 1. Indeed, a quick cross-check of the stationary parts of the numerically obtained moment time histories (as reported here) with the corresponding exact stationary values shows that they match very well. On computing the exact stationary values of first few moments using Eq. (122) (via the symbolic manipulator MAPLE®), we get $\mathbf{E}[x^2] = 0.061382\ldots$, $\mathbf{E}[\dot{x}^2] = 6.250\ldots$, and $\mathbf{E}[x^2\dot{x}^2] = 0.38364\ldots$. The closeness of these values with those

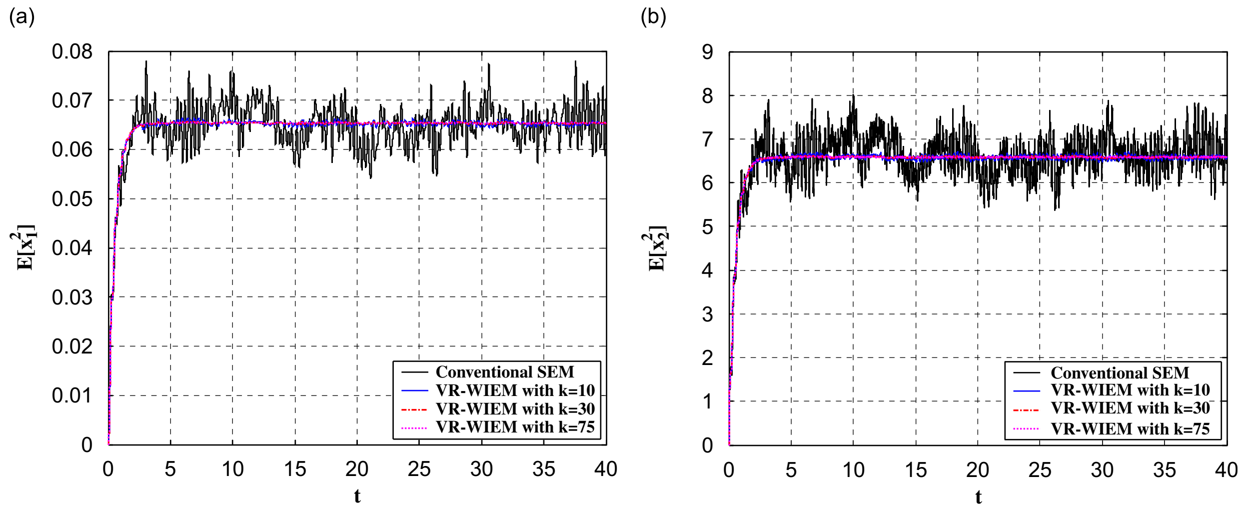

Fig. 7. Moment histories of Duffing equation under additive noise: (a) expectation of displacement quadratic, (b) expectation of velocity quadratic, $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500, all implicitness parameters $= 0.50$.

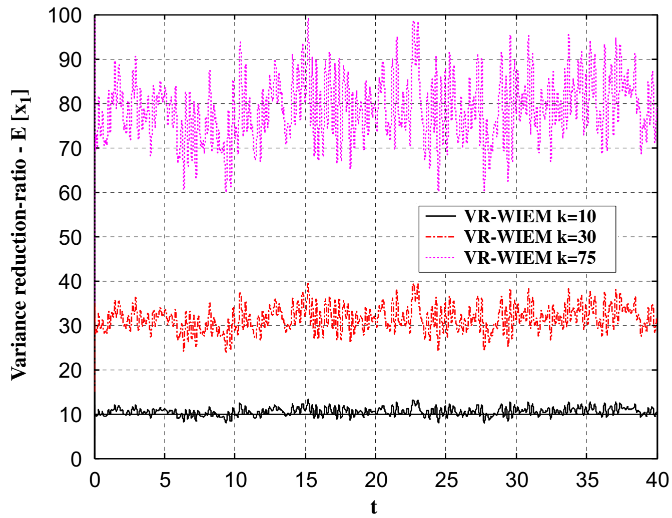

Fig. 8. History of variance reduction ratio of Duffing equation under additive noise for displacement, $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500, all implicitness parameters $= 0.50$.

obtained through weak numerical integration via the variance-reduced algorithm may be readily verified through the figures.

4.1.4. One-sample (deterministic) simulations

Indeed, a major advantage of the present set of formulations is that they may be made nearly deterministic as $k \to \infty$. In other words, one should be able to simulate just one sample with a very high k through any one of the variance-reduced strategies and thus obtain realistic plots of expectation histories. To elaborate upon this point further, we choose the variance-reduced WSNM (VR-WSNM) with $k = 100$ and an ensemble size of just 1. Plots of second- and higher-order moments for the displacement function are provided in Figs. 12(a)–(c) along with comparisons with results obtained through an ensemble size of 500. The closeness of the plots is evident. The variance of expectations in these plots with $k = 100$ comes out close to zero.

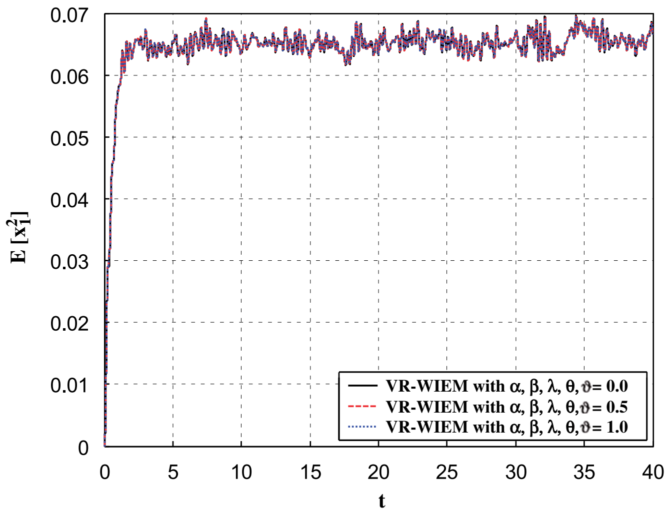

Fig. 9. History of displacement square of Duffing equation under additive noise for various implicitness parameters, $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500.

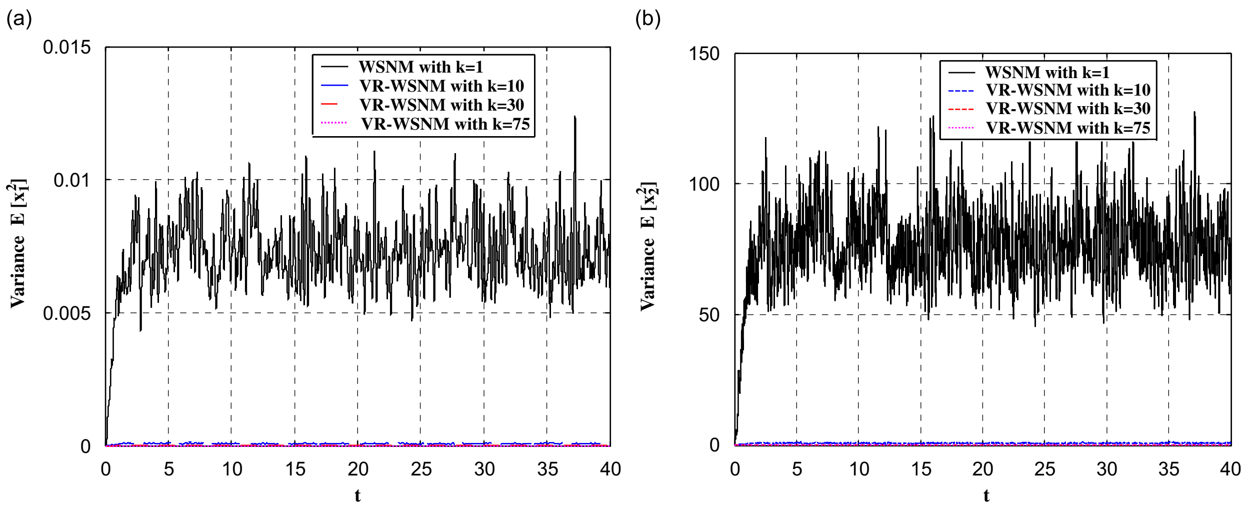

Fig. 10. Variance histories of Duffing equation under additive noise: (a) variance of displacement quadratic, (b) variance of velocity quadratic, $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500, all implicitness parameters = 0.50.

The results clearly show that one-sample simulations may often be adequate for obtaining numerically correct results of expectations of response increments.

4.2. The Duffing equation under additive and multiplicative noises

As a second illustrative example, we consider the Duffing equation subjected to combined additive and multiplicative noises and described by the equation

$$\ddot{x} + C\dot{x} + K_1 x + K_2 x^3 = \sigma_1 \dot{W}_1(t) + \sigma_2 x \dot{W}_2(t). \tag{123}$$

Incidentally, the closed form and exact SPDF for this case is also available. The SPDF is presently of the form [9]

$$p(x, \dot{x}) = A \exp\left(-\frac{C}{\sigma_1^2 + \sigma_2^2 x^2} - \int_0^x C \frac{K_1 s + K_2 s^3}{0.5(\sigma_1^2 + \sigma_2^2 s^2)} \mathrm{d}s\right). \tag{124}$$

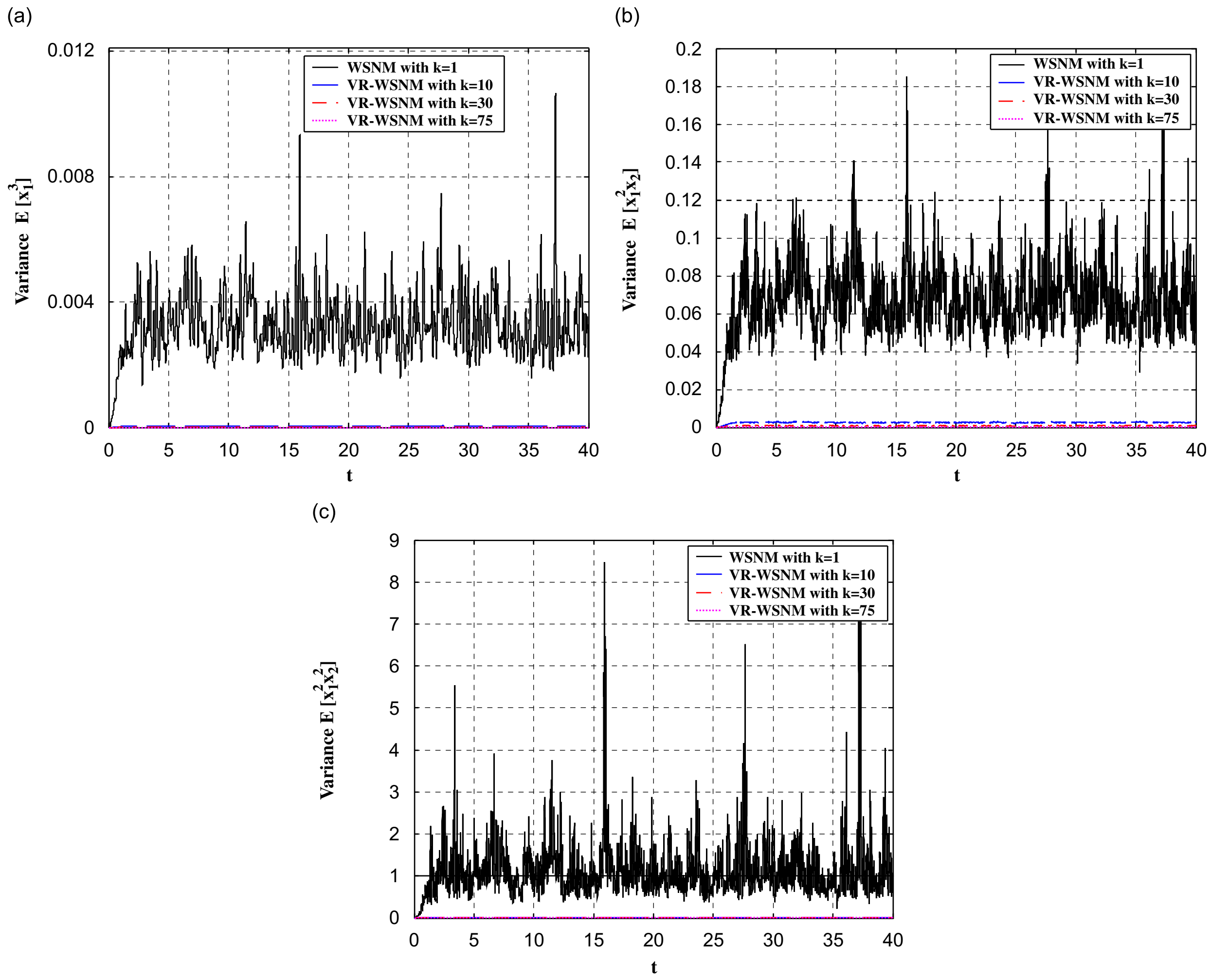

Fig. 11. Variance histories of Duffing equation under additive noise: (a) variance of displacement-cube, (b) variance of displacement-square velocity, (c) variance of displacement-square velocity-square; $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, number of samples: 500, all implicitness parameters $= 0.50$.

Towards demonstrating that the variance reduction method works for combined additive and multiplicative noises as well, we use only the VR-WSNM as a representative scheme with $k = 100$ and an ensemble size of just 1. In order to further emphasize the performance of the VR-WSNM, we plot in Fig. 13 the error history as measured against the exact analytical solution (in the stationary limit). The error e_f, given a scalar function $f(x, \dot{x})$, is presently defined as the absolute value of the ratio

$$e_f = \left| \frac{\mathbf{E}^A[f(x,\dot{x})] - \mathbf{E}^N[f(x,\dot{x})]}{\mathbf{E}^A[f(x,\dot{x})]} \right| \quad \text{provided } \mathbf{E}^A[f(x,\dot{x})] \neq 0, \tag{125}$$

where $\mathbf{E}^A$ is the expectation corresponding to the exact stationary solution and $\mathbf{E}^N$ is that obtained through a numerical method. In Figs. 13(a) and (b), we have, respectively, taken $f = x^2$ and $f = \dot{x}^2$, which are strictly positive functions for nonzero x and $\dot{x}$. Thus, the above definition makes sense. It is evident that the results obtained by VR-WSNM have far better correspondence with exact solutions than those via the conventional WSNM.

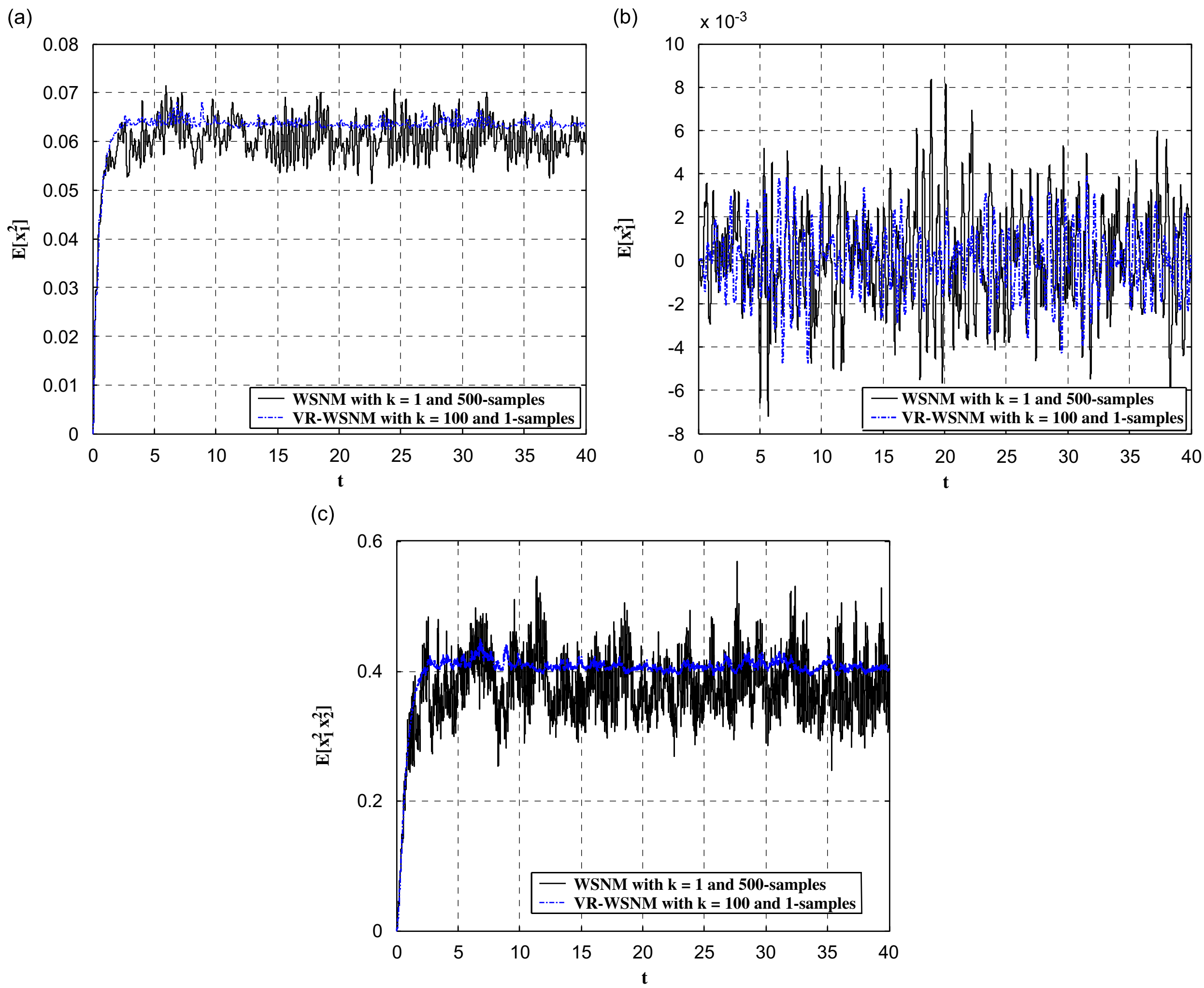

Fig. 12. Moment histories of Duffing equation under additive noise: (a) expectation of displacement-square, (b) expectation of displacement-cube, (c) expectation of displacement-square velocity-square, $C = 2.0$, $K_1 = 100$, $K_2 = 10$, $\sigma = 5.0$, $h = 0.001$, all implicitness parameters $= 0.50$.

4.3. Mdof systems: 2- and 3-dof nonlinear oscillators

We presently attempt to demonstrate the ready applicability of the proposed variance reduction method for higher-dimensional problems, i.e., mdof oscillators. In particular, we choose a couple of 2- and 3-dof oscillators with polynomial nonlinearity. The governing equations, under purely additive noises, are presently given by

$$\begin{aligned} &\ddot{x}_1 + C_1\dot{x}_1 + (K_1 + K_2)x_1 - K_2x_3 + \alpha_1 x_1^3 = \sigma_1 \dot{W}_1(t), \\ &\ddot{x}_3 + C_2\dot{x}_3 - K_2x_1 + K_2x_3 + \alpha_2 x_3^3 = \sigma_2 \dot{W}_2(t) \end{aligned} \tag{126}$$

and

$$\begin{aligned} &\ddot{x}_1 + C_1\dot{x}_1 + (K_1 + K_2)x_1 - K_2x_3 + \alpha_1 x_1^3 = \sigma_1 \dot{W}_1(t), \\ &\ddot{x}_3 + C_2\dot{x}_3 - K_2x_1 + (K_2 + K_3)x_3 - K_3x_5 + \alpha_2 x_3^3 = \sigma_2 \dot{W}_2(t), \\ &\ddot{x}_5 + C_3\dot{x}_5 - K_3x_3 + K_3x_5 + \alpha_3 x_5^3 = \sigma_3 \dot{W}_3(t) \end{aligned} \tag{127}$$

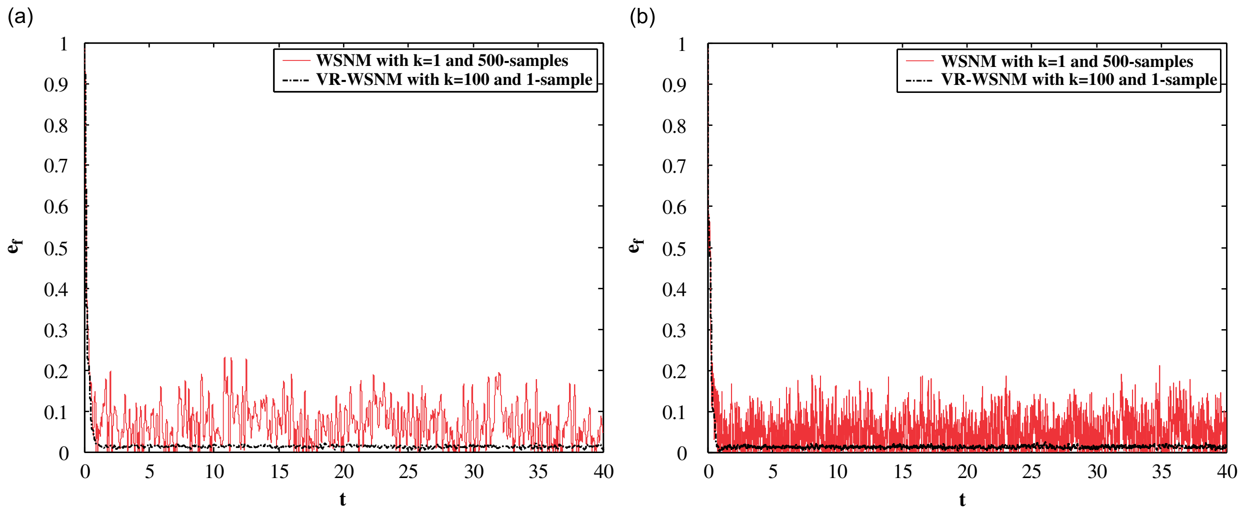

Fig. 13. Error histories of Duffing equation under additive and multiplicative noise: (a) error in expectation of displacement-square, (b) error in expectation of velocity-square; $C = 5.0$, $K_1 = 100$, $K_2 = 10$, $\sigma_1 = 5.0$, $\sigma_2 = 5.0$, $h = 0.001$, number of samples: 500, all implicitness parameters $= 0.50$.

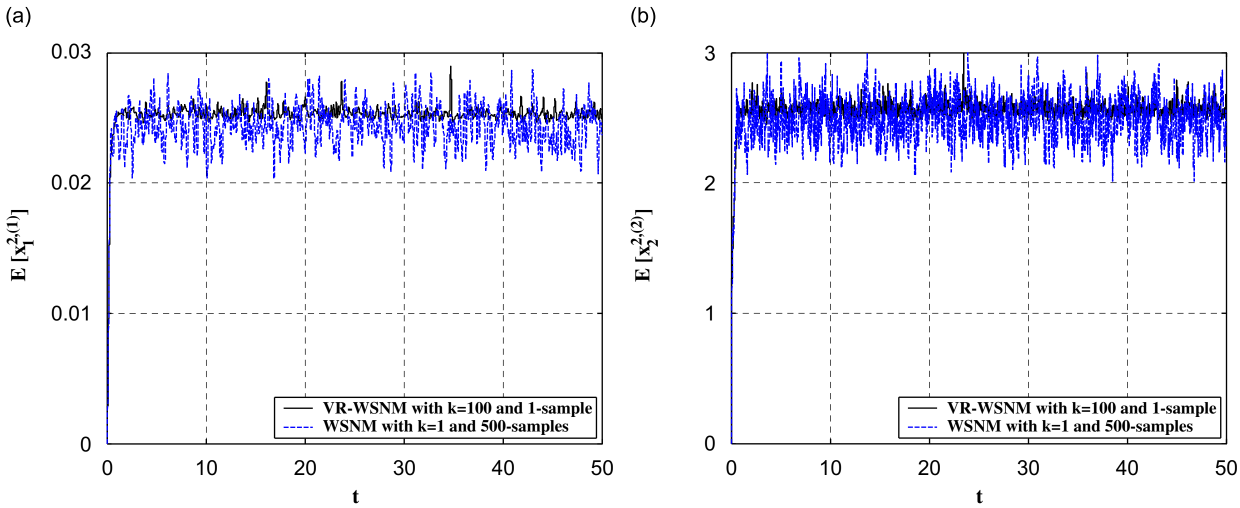

Fig. 14. Second moment histories of 2-dof nonlinear oscillator under additive noise: (a) $\mathbf{E}[x_1^{2,(1)}]$, i.e., expectation of 1-dof displacement quadratic; (b) $\mathbf{E}[x_2^{2,(2)}]$, i.e., expectation of 2-dof velocity quadratic; $C_1 = C_2 = 2.0$; $K_1 = K_2 = 100.0$; $\alpha_1 = \alpha_2 = 10.0$; $\sigma_1 = \sigma_2 = 2.0$; $h = 0.001$.

for the 2- and 3-dof oscillators, respectively. Now to be consistent with the general formulation (as in Section 2 followed by further illustrations in this section for the sdof oscillator), we write the VR-WSNM maps for the 2-dof oscillator in terms of the renamed variables $\left\{x_1^{(j)}, x_2^{(j)} | j \in [1,2]\right\}$ so that $x_1^{(1)} := x_1$ and $x_1^{(2)} := x_3$. In order to compute the expectations with accuracy up to $O(h^2)$, we also need additional maps to determine the quadratic terms $\left\{x_1^{2,(1)}, x_2^{2,(1)}, x_1^{2,(2)}, x_2^{2,(2)}\right\}$ and the cross-quadratic terms $\left\{x_1^{(1)}x_2^{(1)}, x_1^{(2)}x_2^{(2)}, x_1^{(1)}x_1^{(2)}, x_1^{(1)}x_2^{(2)}, x_2^{(1)}x_1^{(2)}, x_2^{(1)}x_2^{(2)}\right\}$. This would result in 14 VR-WSNM maps. In order to avoid too many implicitness parameters associated with the VR-WSNM maps, we use the explicit VR-WEM strategy for the 3-dof oscillator. Indeed, for higher-dof oscillators, we prescribe a cautious use of the implicit VR-WSNM owing to the multiplicity of possible solutions (while solving for the zeros of the nonlinear maps over each step through a Newton–Raphson or a fixed point technique). Presently, we write the explicit VR-WEM maps (precisely as described for the general formulation in Section 2 with $n = 3$) in terms of the variables $\left\{x_1^{(j)}, x_2^{(j)} | j \in [1,3]\right\}$ and their quadratic and cross-quadratic terms. Using $k = 100$ and the smallest possible ensemble size of just 1, the SDEs for the 2-dof oscillator are integrated using the VR-WSNM and those for the 3-dof oscillator are integrated via the VR-WEM. Plots of a few second moment histories of some displacement and velocity components are shown in Figs. (14) and (15) for the 2- and 3-dof cases, respectively. Since, there are presently no closed-form solutions available, comparisons of the moment histories are made with direct MCSs through the weak forms of the methods without variance reduction. From the figures it is amply clear that the results with and without variance reduction are in close correspondence with the anticipated exception that the variances of estimated moments with variance reduction are significantly lower. The authors would like to add that the mdof oscillators, given by SDEs (126) and (127), are presently chosen as merely representative examples and not with any specific engineering application in mind.

It is readily understandable that the proposed Variance-reduced weak simulation strategy is far more computationally efficient vis-à-vis a direct MCS. To provide some idea of the relative computational

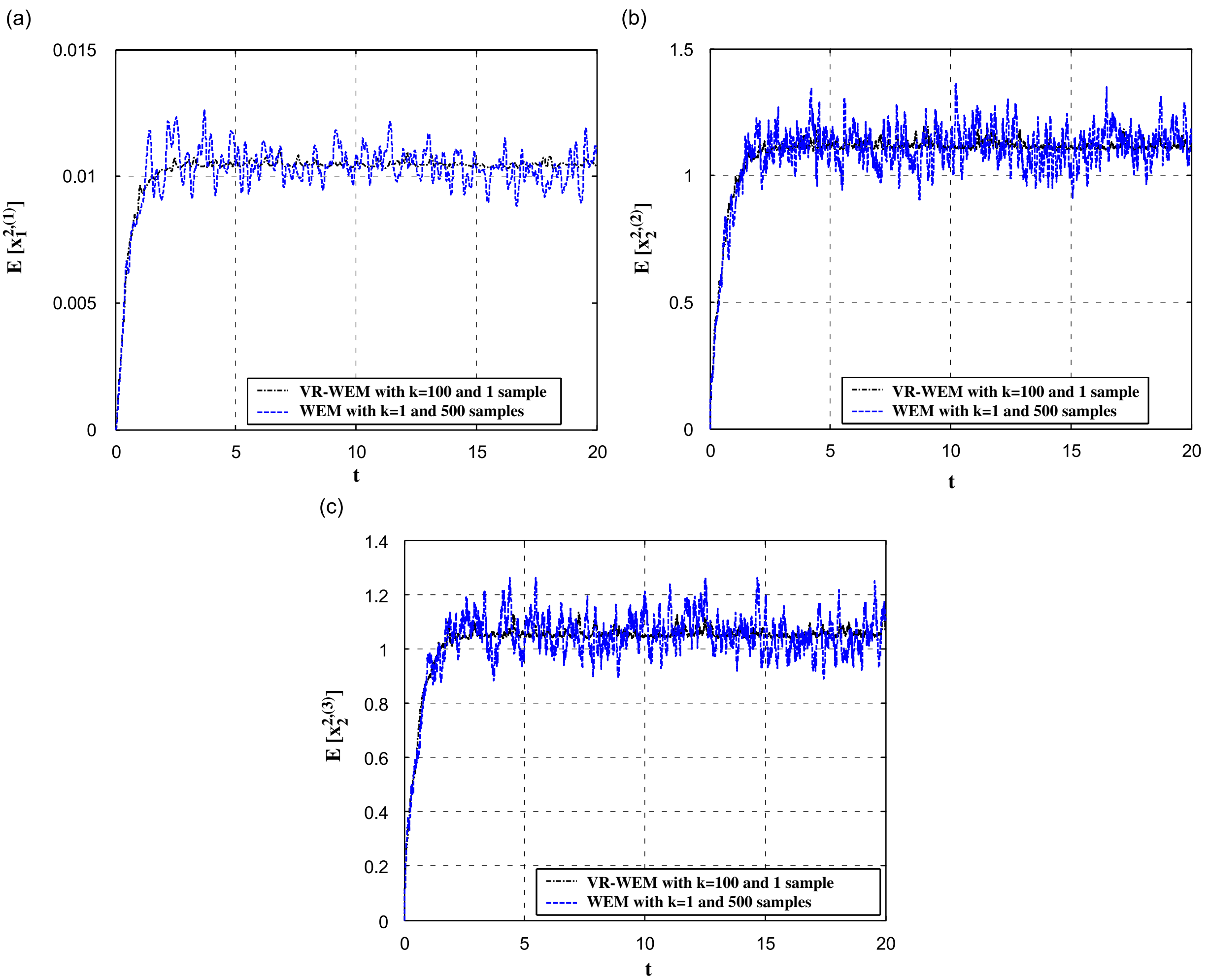

Fig. 15. Second moment histories of 3-dof nonlinear oscillator under additive noise: (a) $\mathbf{E}[x_1^{2,(1)}]$, i.e., expectation of 1-dof displacement quadratic; (b) $\mathbf{E}[x_2^{2,(2)}]$, i.e., expectation of 2-dof velocity quadratic; (c) $\mathbf{E}[x_2^{2,(3)}]$, i.e., expectation of 3-dof velocity quadratic; $C_1 = C_2 = C_3 = 2.0$; $K_1 = K_2 = K_3 = 100.0$; $\alpha_1 = \alpha_2 = \alpha_3 = 10.0$; $\sigma_1 = \sigma_2 = \sigma_3 = 2.0$; $h = 0.001$.

Table 1
A comparison of CPU times via VR-WEM (ensemble size 1 and $k = 100$) and direct WEM (ensemble size 5000) over a time interval 0–20 s for various nonlinear oscillators: (Eq. (127) $C_1 = C_2 = C_3 = 2.0$; $K_1 = K_2 = K_3 = 100.0$; $\alpha_1 = \alpha_2 = \alpha_3 = 10.0$; $\sigma_1 = \sigma_2 = \sigma_3 = 2.0$; $h = 0.001$, Eq. (126) $C_1 = C_2 = 2.0$; $K_1 = K_2 = 100.0$; $\alpha_1 = \alpha_2 = 10.0$; $\sigma_1 = \sigma_2 = 2.0$; $h = 0.001$, Eq. (112) $C = 2.0$; $K_1 = 100.0$; $K_2 = 10.0$; $\sigma = 2.0$; $h = 0.001$)

DOF type	CPU time required (s)	
	VR-WEM	Direct WEM
3-dof (Eq. (127))	318	4200
2-dof (Eq. (126))	219	2910
1-dof (Eq. (112))	110	1450

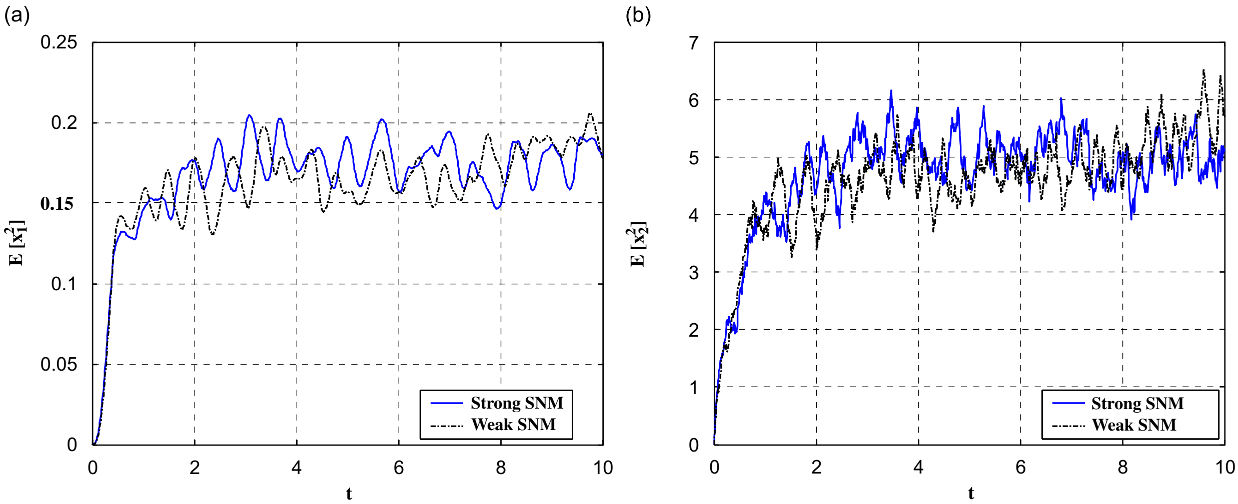

Fig. 16. Second moment histories of Duffing equation under additive noise: (a) expectation of displacement quadratic, (b) expectation of velocity quadratic; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 0.50$, $\varepsilon_3 = 0.0$, $\varepsilon_4 = 0.10$, $h = 0.01$, number of samples: 200.

advantage, Table 1 shows a typical comparison of CPU times required via VR-WEM (using an ensemble size 1 and $k = 100$) and direct WEM (using an ensemble size of 5000) for oscillators of different dofs. The computer used for simulations is an Intel® P4, 2.4 GHz machine with 1 GB RAM. It is interesting to note that, even with 5000 samples in the ensemble, direct simulations (without variance reduction) generally produce much higher variances in the expectations than is achievable through simulations with variance reduction.

Before concluding this section, the authors find it relevant to mention the following point. In a previous paper [19], it was claimed (Proposition 4, Section 3) that the moment equations are satisfied by a distribution $P(\lambda_r = \pm\sqrt{3}) = 1/6$, $P(\lambda_r = 0) = 2/3$ and $\eta_r = 0.5\lambda_r$. However, we have detected an inconsistency with the above distribution and have accordingly modified the distribution as in Eq. (28), which is recast in the following form in such a way that the distribution becomes independent of h (to be consistent with [19])

$$P(\lambda_r = \pm 1) = 1/2 \quad \text{and} \quad \eta_r = 0.5\lambda_r + \gamma_r \quad \text{and} \quad P(\gamma_r = \pm\sqrt{1/12}) = 1/2. \tag{128}$$

Here, γ_r and λ_r are independently generated random variables. To show that our distribution works well, we plot the second moment histories of displacement and velocity in Figs. 16(a) and (b) via the conventional form of WSNM (i.e., without any augmentation of the state variables). In the same figure, we also show the comparison of these plots with the results obtained via strong form of SNM. The ensemble size is chosen to be 200 (same as in Ref. [19]) with no externally applied deterministic force. The values taken are analogous to the results shown for strong additive noise. The Duffing equation is also recast in the same form as in Roy [19]:

$$\begin{aligned} \mathrm{d}x_1 &= x_2 \mathrm{d}t, \\ \mathrm{d}x_2 &= (-2\pi\varepsilon_1 x_2 - 4\pi^2\varepsilon_2(1 + x_1^2)x_1 + 4\pi^2\varepsilon_3 \cos(2\pi t))\,\mathrm{d}t + 4\pi^2\varepsilon_4\,\mathrm{d}W_1(t). \end{aligned} \tag{129}$$

It may be observed that fluctuations are generally higher in weak solutions when compared with strong solutions.

5. Conclusions

We have proposed and explored a generally applicable weak principle for variance-reduced response simulations of mechanical oscillators driven by additive and/or multiplicative white noise processes. The well-known method of achieving a reduction in variance (of solutions of SDEs) is through an optimal modification of the drift term (which is a consequence of Girsanov-type change of the underlying probability measures) and the construction of such modified drift terms, even in sub-optimal cases, requires an a priori knowledge of statistical quantities of interest and their derivatives with respect to the initial conditions [16,22,25]. In Ref. [25], for instance, the expectation (to be found using a variance-reduced simulation) is approximately determined through a numerical solution of the associated partial differential equation (i.e., the backward Kolmogorov equation). The present formulation, on the other hand, does not need any such a priori information and can, in principle, simultaneously obtain variance-reduced estimates of expectations any set of function of the response variables provided that such functions admit Taylor expansions in terms of the response variables. The key to this formulation is to augment the existing set of response variables by an additional set of variables consisting of different powers of elements of the existing set. The next step is to form an augmented set of equations of motions (in the form of SDEs) and identify the stochastic terms in each of these equations are then identified. Since these augmented equations need to be integrated over a small step-size, h, the final step is to statistically characterize the stochastic terms through their first few moments, that are all expressible in various powers of h, and subsequently replace these stochastic terms weakly through a set of statistically equivalent stochastic terms (up to the same order of h) with reduced variances. If these stochastic terms appear purely due to a set of additive noises, then the terms are (locally) independent of the response variables and this enables us to reduce the variance by any desired factor. An immediate consequence of this observation is that it is possible to numerically compute near-deterministic expectations of the required quantities even with a simulation of just one realization of the augmented response processes. Indeed, one-sample simulations with a very high variance reduction factor should provide a way out of the well known problem of moment closure in nonlinear dynamical systems. While the methodology is general enough, we have applied it in the context of weak stochastic simulations via Euler, implicit Euler and Newmark methods. We have also provided an analysis of the local rates of convergence of these variance-reduced methods. Finally, we have considered a limited numerical illustration of the procedures through their applications to a few sdof and mdof nonlinear oscillators under additive and/or multiplicative excitations. Comparisons with exact solutions, whenever available, are also provided. A more detailed numerical exploration of this method is presently under way for oscillators with other forms of nonlinearity.

It is of interest to note that the extra computational price paid in the form of an augmentation of the system equations is generally overshadowed by the advantages accrued owing to a drastic reduction of ensemble size. Assuming for an n-dof system an ensemble size of 5000 samples to be good enough for the statistical estimation of the first few moments via Euler's method (SEM), the total number of scalar equations to be integrated in the state space is $2n \times 5000$. On the other hand, the number of such equations required to be integrated via the variance-reduced Euler method with an ensemble size of just 1 is $2n$ (for mean equations) + $2n$ (for square equations) $+{}^{2n}C_2$ (for calculating cross-moments). This implies the computational advantages of the Variance-reduced simulations continue over classical simulations as long as system dofs remain less than 5000. Similar arguments would hold for variance-reduced WIEM and WSNM.

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Appendix A

Let us use the following notations for simplicity of writing the expressions

$$C_{i-1} := C(\bar{X}_{i-1}, t_{i-1}), \quad K_{i-1} := K(\bar{X}_{i-1}, t_{i-1}), \quad \sigma_r = \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1}).$$

Though the following equations have a large number of terms on the RHS, they may readily be obtained through a symbolic manipulator, such as MAPLE®:

$$\begin{aligned} x_{1,i}^{(j)} = {} & x_{1,i-1} + x_{2,i-1}h + \sum_{r=1}^{q} \sigma_r I_{r0} + 0.5\alpha a^{(j)}(\bar{X}_{i-1}, t_{i-1})h^2 \\ & + 0.5(1-\alpha)a^{(j)}(\bar{X}_i, t_i)h^2 + R_{\mathrm{SNM}.d}^{(j)} \quad j \in [1, n], \end{aligned} \tag{A.1}$$

$$\begin{aligned} x_{2,i}^{(j)} = {} & x_{2,i-1}^{(j)} + \sum_{r=1}^{q} \sigma_r^{(j)}(\bar{X}_{i-1}, t_{i-1})I_r + \beta a^{(j)}(\bar{X}_{i-1}, t_{i-1})h \\ & + (1-\beta)a^{(j)}(\bar{X}_i, t_i)h + R_{\mathrm{SNM}.v}^{(j)} \quad j \in [1, n], \end{aligned} \tag{A.2}$$

$$\begin{aligned} x_{1,i}^{(j)}x_{1,i}^{(k)} = {} & [x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} + x_{1,i-1}^{(j)} \sum_{l=1}^{q} \sigma_l^{(k)} I_{l0} + x_{1,i-1}^{(k)} \sum_{r=1}^{q} \sigma_r^{(j)} I_{r0} + \sum_{r,l=1}^{q} \sigma_r^{(j)}\sigma_l^{(k)} I_{r0} I_{l0}] \\ & + [x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{1,i-1}^{(k)}x_{2,i-1}^{(j)} + x_{2,i-1}^{(j)} \sum_{l=1}^{q} \sigma_l^{(k)} I_{l0} + x_{2,i-1}^{(k)} \sum_{r=1}^{q} \sigma_r^{(j)} I_{r0}]h \\ & + [x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} - K_{i-1}x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} - 0.5K_{i-1}x_{1,i-1}^{(j)} \sum_{l=1}^{q} \sigma_l^{(k)} I_{l0} - 0.5K_{i-1}x_{1,i-1}^{(k)} \sum_{r=1}^{q} \sigma_r^{(j)} I_{r0} \\ & - 0.5C_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} - 0.5C_{i-1}x_{1,i-1}^{(k)}x_{2,i-1}^{(j)} - 0.5C_{i-1}x_{2,i-1}^{(j)} \sum_{l=1}^{q} \sigma_l^{(k)} I_{l0} \\ & - 0.5C_{i-1}x_{2,i-1}^{(k)} \sum_{r=1}^{q} \sigma_r^{(j)} I_{r0} + 0.5F_{i-1}(t)x_{1,i-1}^{(j)} + 0.5F_{i-1}(t)x_{1,i-1}^{(k)} \\ & + 0.5F_{i-1}(t) \sum_{r=1}^{q} \sigma_r^{(j)} I_{r0} + 0.5F_{i-1}(t) \sum_{l=1}^{q} \sigma_l^{(k)} I_{l0}]h^2 \\ & + [-C_{i-1}x_{2,i-1}^{(k)}x_{2,i-1}^{(j)} - 0.5K_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} - 0.5K_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + F_{i-1}(t)x_{2,i-1}^{(j)}]h^3 \\ & + \chi[0.25C_{i-1}K_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + 0.25C_{i-1}K_{i-1}x_{1,i-1}^{(k)}x_{2,i-1}^{(j)} + 0.25K_{i-1}^2 x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} \\ & + 0.25C_{i-1}^2 x_{2,i-1}^{2,(j)} - 0.25F_{i-1}(t)C_{i-1}x_{2,i-1}^{(j)} - 0.25F_{i-1}(t)C_{i-1}x_{2,i-1}^{(k)} \\ & - 0.25F_{i-1}(t)K_{i-1}x_{1,i-1}^{(j)} - 0.25F_{i-1}(t)K_{i-1}x_{1,i-1}^{(k)}]h^4 \\ & + (1-\chi)[0.25C_iK_ix_{1,i}^{(j)}x_{2,i}^{(k)} + 0.25C_iK_ix_{1,i}^{(k)}x_{2,i}^{(j)} + 0.25K_i^2 x_{1,i}^{(j)}x_{1,i}^{(k)} \\ & + 0.25C_i^2 x_{2,i}^{2,(j)} - 0.25F_i(t)C_ix_{2,i}^{(j)} - 0.25F_i(t)C_ix_{2,i}^{(k)} \\ & - 0.25F_i(t)K_ix_{1,i}^{(j)} - 0.25F_i(t)K_ix_{1,i}^{(k)}]h^4 + R_{\mathrm{SNM},d^2}^{(j,k)} \quad (j,k) \in [1,n] \times [1,n], \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
x_{2,i}^{(j)}x_2^{(k)} &= x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} + \sum_{r=1}^{q}\sigma_r^{(j)}(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(k)}I_r + \sum_{l=1}^{q}\sigma_l^{(k)}(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(j)}I_l \\
&+ \sum_{r,l=1}^{q}\sigma_r^{(j)}\sigma_l^{(k)}(\bar{X}_{i-1},t_{i-1})I_rI_l + [-2C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} \\
&- K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} - K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(k)}x_{2,i-1}^{(j)}]h \\
&+ \sum_{r=1}^{q}\sigma_r^{(j)}(\bar{X}_{i-1},t_{i-1})[-C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(k)} - K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(k)} + F(t)]hI_r \\
&+ \sum_{l=1}^{q}\sigma_l^{(k)}(\bar{X}_{i-1},t_{i-1})[-C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(j)} - K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(j)} + F(t)]hI_l \\
&+ \theta[C(\bar{X}_{i-1},t_{i-1})^2x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} + K(\bar{X}_{i-1},t_{i-1})^2x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} + F^2(t) \\
&+ C(\bar{X}_{i-1},t_{i-1})K(\bar{X}_{i-1},t_{i-1})[x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{1,i-1}^{(k)}x_{2,i-1}^{(j)}] \\
&- F(t)\{C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(j)} + K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(j)}\} \\
&- F(t)\{C(\bar{X}_{i-1},t_{i-1})x_{2,i-1}^{(k)} + K(\bar{X}_{i-1},t_{i-1})x_{1,i-1}^{(k)}\}]h^2 \\
&+ (1-\theta)[C(\bar{X}_i,t_i)^2x_{2,i}^{(j)}x_{2,i}^{(k)} + K(\bar{X}_i,t_i)^2x_{1,i}^{(j)}x_{1,i}^{(k)} + F^2(t) \\
&+ C(\bar{X}_i,t_i)K(\bar{X}_i,t_i)[x_{1,i}^{(j)}x_{2,i}^{(k)} + x_{1,i}^{(k)}x_{2,i}^{(j)}] \\
&- F(t)\{C(\bar{X}_i,t_i)x_{2,i}^{(j)} + K(\bar{X}_i,t_i)x_{1,i}^{(j)}\} \\
&- F(t)\{C(\bar{X}_i,t_i)x_{2,i}^{(k)} + K(\bar{X}_i,t_i)x_{1,i}^{(k)}\}]h^2 + R_{\mathrm{SNM},v^2}^{(j,k)} \quad (j,k)\in[1,n]\times[1,n], \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
x_{1,i}^{(j)}x_{2,i}^{(k)} &= [x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{1,i-1}^{(j)}\sum_{l=1}^{q}\sigma_l^{(k)}I_l + x_{2,i-1}^{(k)}\sum_{r=1}^{q}\sigma_r^{(j)}I_{r0} + \sum_{l=1}^{q}\sum_{r=1}^{q}\sigma_r^{(j)}\sigma_l^{(k)}I_rI_{l0}] \\
&+ [-C_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} - K_{i-1}x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} - C_{i-1}x_{2,i-1}^{(k)}\sum_{r=1}^{q}\sigma_r^{(j)}I_{r0} - K_{i-1}x_{1,i-1}^{(k)}\sum_{r=1}^{q}\sigma_r^{(j)}I_{r0} \\
&+ x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} + x_{2,i-1}^{(j)}\sum_{l=1}^{q}\sigma_l^{(k)}I_l + x_{1,i-1}^{(j)}F_{i-1}(t) + F_{i-1}(t)\sum_{r=1}^{q}\sigma_r^{(j)}I_{r0}]h \\
&+ [-1.5C_{i-1}x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} - 1.5K_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} - 0.5K_{i-1}x_{1,i-1}^{(j)}\sum_{l=1}^{q}\sigma_l^{(k)}I_l \\
&- 0.5C_{i-1}x_{2,i-1}^{(j)}\sum_{l=1}^{q}\sigma_l^{(k)}I_l + 0.5F_{i-1}(t)\sum_{l=1}^{q}\sigma_l^{(k)}I_l + 1.5F_{i-1}(t)x_{2,i-1}^{(k)}]h^2 \\
&+ \vartheta[C_{i-1}K_{i-1}x_{1,i-1}^{(j)}x_{2,i-1}^{(k)} + 0.5K_{i-1}^2x_{1,i-1}^{(j)}x_{1,i-1}^{(k)} + 0.5C_{i-1}^2x_{2,i-1}^{(j)}x_{2,i-1}^{(k)} \\
&+ 0.5F_{i-1}^2(t) - F_{i-1}(t)C_{i-1}x_{2,i-1}^{(j)} - F_{i-1}(t)K_{i-1}x_{1,i-1}^{(j)}]h^3 \\
&+ (1-\vartheta)[C_iK_ix_{1,i}^{(j)}x_{2,i}^{(k)} + 0.5K_i^2x_{1,i}^{(j)}x_{1,i}^{(k)} + 0.5C_i^2x_{2,i}^{(j)}x_{2,i}^{(k)} \\
&+ 0.5F_i^2(t) - F_i(t)C_ix_{2,i}^{(j)} - F_i(t)K_ix_{1,i}^{(j)}]h^3 + R_{\mathrm{SNM},dv}^{(j,k)} \quad (j,k)\in[1,n]\times[1,n]. \qquad \text{(A.5)}
\end{aligned}
$$

Appendix B. Modelling of random variables

Stochastic modelling of the some of the MSIs, viz., I_r^2; I_{r0}^2 and I_rI_{r0}, are derived in detail while modelling for others may be performed on similar lines.

B.1. Modelling of $z = I_r^2$

$$\text{Let } \mathrm{d}x(t) = \mathrm{d}W(t) \text{ so that } x(t) = \int_0^t \mathrm{d}W(t). \tag{B.1}$$

Then we get $z(s) = x^2(s)$, i.e., $z = I_r^2$.

B.1.1. Calculating the mean of z(t) i.e., $\mathbf{E}(z(t))$*:*

$$\mathbf{E}(z(t)) = \mathbf{E}(x^2(s)). \tag{B.2}$$

Dropping the coefficients of 't':

$$\mathrm{d}(x^2) = (1)\mathrm{d}t + (\cdots)\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(x^2) = \int_0^t \mathrm{d}s, \tag{B.3}$$

$$\mathbf{E}(x^2) = \int_0^t \mathrm{d}s = t. \tag{B.4}$$

B.1.2. Calculating the second moment of z(t), i.e., $\mathbf{E}(z^2(t)) = \mathbf{E}(I_r^2)$*:*

$$\mathrm{d}(z^2) = \mathrm{d}(x^4) = (6x^2)\,\mathrm{d}t + (\cdots)\,\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(x^4) = 6\int_0^t \mathbf{E}(x^2)\,\mathrm{d}s. \tag{B.5}$$

Using Eq. (B.4),

$$\mathbf{E}(x^4) = 6\int_0^t s\,\mathrm{d}s = 3t^2. \tag{B.6}$$

This means that $I_r^2 \sim N(t, 3t^2)$.

B.2. Modelling of $z = I_{r0}^2$

$$\text{Again, } \mathrm{d}x(t) = \mathrm{d}W(t) \quad x(t) = \int_0^t \mathrm{d}W(t), \tag{B.7}$$

$$\mathrm{d}y(t) = x(t)\,\mathrm{d}t \quad y(t) = \int_0^t x(s)\,\mathrm{d}s = \int_0^t \int_0^s \mathrm{d}W(s_1)\,\mathrm{d}s. \tag{B.8}$$

Then we get, $z(s) = y^2(s)$.

B.2.1. Calculating the mean of z(t) i.e., $\mathbf{E}(z(t)) = \mathbf{E}(I_{r0}^2)$

Dropping the coefficients of 't',

$$\mathrm{d}(y^2) = (2xy)\mathrm{d}t \quad \Rightarrow \mathbf{E}(y^2) = 2\int_0^t \mathbf{E}(xy)\mathrm{d}s, \tag{B.9}$$

$$\mathrm{d}(xy) = (x^2)\,\mathrm{d}t + (\cdots)\,\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(xy) = \int_0^t \mathbf{E}(x^2)\,\mathrm{d}s. \tag{B.10}$$

Using Eq. (B.4),

$$\mathbf{E}(xy) = \int_0^t \mathbf{E}(x^2)\,\mathrm{d}s = \frac{t^2}{2}. \tag{B.11}$$

Using Eq. (B.11) in Eq. (B.9),

$$\mathbf{E}(y^2) = 2\int_0^t \mathbf{E}(xy)\,\mathrm{d}s = 2\int_0^t \frac{s^2}{2} = \frac{t^3}{3}. \tag{B.12}$$

B.2.2. Calculating the second moment of z(t), i.e., $\mathbf{E}(z^2(t))$

$$\mathrm{d}(z^2) = \mathrm{d}(y^4) = (4y^3x)\mathrm{d}t + (\cdots)\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(y^4) = 4\int_0^t \mathbf{E}(y^3x)\,\mathrm{d}s, \tag{B.13}$$

$$\mathrm{d}(xy^3) = (3y^2x^2)\mathrm{d}t \quad \Rightarrow \mathbf{E}(xy^3) = \int_0^t \mathbf{E}(3y^2x^2)\mathrm{d}s, \tag{B.14}$$

$$\mathrm{d}(y^2x^2) = (2yx^3 + y^2)\mathrm{d}t + (\cdots)\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(y^2x^2) = \int_0^t \mathbf{E}(2yx^3 + y^2)\mathrm{d}s, \tag{B.15}$$

$$\mathrm{d}(yx^3) = (x^4 + 3yx)\mathrm{d}t + (\cdots)\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(yx^3) = \int_0^t \mathbf{E}(x^4 + 3yx)\mathrm{d}s, \tag{B.16}$$

$$\mathbf{E}(yx^3) = \int_0^t \mathbf{E}(x^4 + 3yx)\mathrm{d}s = \int_0^t \left(3s^2 + 3\frac{s^2}{2}\right)\mathrm{d}s = \frac{3t^3}{2}, \tag{B.17}$$

$$\mathbf{E}(y^2x^2) = \int_0^t \mathbf{E}(2yx^3 + y^2)\mathrm{d}s = \int_0^t \left(2\frac{3s^3}{2} + \frac{s^3}{3}\right)\mathrm{d}s = \frac{10}{3}\frac{t^4}{4} = \frac{5t^4}{6}, \tag{B.18}$$

$$\mathbf{E}(xy^3) = \int_0^t \mathbf{E}(3y^2x^2)\mathrm{d}s = \int_0^t 3\frac{5s^4}{6}\mathrm{d}s = \frac{t^5}{2}, \tag{B.19}$$

$$\mathbf{E}(y^4) = \int_0^t \mathbf{E}(4xy^3)\,\mathrm{d}s = \int_0^t \left(4\frac{s^5}{2}\right)\mathrm{d}s = \frac{t^6}{3}. \tag{B.20}$$

This means that $I_{r0}^2 \sim N\left(\frac{t^3}{3}, \frac{t^6}{3}\right)$

B.3. Modelling of $z = I_r I_{r0}$

$$\text{Again, } \mathrm{d}x(t) = \mathrm{d}W(t) \quad x(t) = \int_0^t \mathrm{d}W(t), \tag{B.21}$$

$$\mathrm{d}y(t) = x(t)\mathrm{d}t \quad y(t) = \int_0^t x(s)\mathrm{d}s = \int_0^t \int_0^s \mathrm{d}W(s_1)\mathrm{d}s, \tag{B.22}$$

then we get $z(s) = x(s)y(s)$.

B.3.1. Calculating the mean of z(t) i.e., $\mathbf{E}(z(t)) = \mathbf{E}(I_r I_{r0})$

$$\mathrm{d}(xy) = (x^2)\mathrm{d}t + (\cdots)\,\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(xy) = \int_0^t \mathbf{E}(x^2)\,\mathrm{d}s. \tag{B.23}$$

Using Eq. (B.4),

$$\mathbf{E}(xy) = \int_0^t \mathbf{E}(x^2)\,\mathrm{d}s = \frac{t^2}{2}. \tag{B.24}$$

B.3.2. Calculating the second moment of z(t), i.e., $\mathbf{E}(z^2(t))$

$$\mathrm{d}(y^2x^2) = (2yx^3 + y^2)\,\mathrm{d}t + (\cdots)\,\mathrm{d}W(t) \quad \Rightarrow \mathbf{E}(y^2x^2) = \int_0^t \mathbf{E}(2yx^3 + y^2)\,\mathrm{d}s. \tag{B.25}$$

Using Eq. (B.18)

$$\mathbf{E}(y^2x^2) = \int_0^t \mathbf{E}(2yx^3 + y^2)\,\mathrm{d}s = \int_0^t \left(2\frac{3s^3}{2} + \frac{s^3}{3}\right)\mathrm{d}s = \frac{10}{3}\frac{t^4}{4} = \frac{5t^4}{6}.$$

This means that $I_r I_{r0} \sim N\left(\frac{t^2}{2}, \frac{5t^4}{6}\right)$.

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