A Monte Carlo approach for efficient estimation of extreme response statistics

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Introduction

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- Are standard Monte Carlo methods really useless in this context?

- NOT QUITE!
The Response Process

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where:
- \( M \) denotes a generalized \( n \times n \) mass matrix,
- \( X = X(t) = (X_1(t), \ldots, X_n(t))^T \) is the system response vector,
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- Hence the solution \( X(t) \) is also a stochastic vector process.

- For specific prediction purposes, it is usually the extreme values of one, or possibly a combination of several, of the component processes of \( X(t) \) that is sought. For simplicity, denote it by \( X(t) \).
The Mean Upcrossing Rate

\[ N^+(\xi; t_1, t_2) = \text{the random number of times that the process } X(t) \text{ upcrosses the level } \xi \text{ during a time interval } (t_1, t_2). \]
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Under suitable regularity conditions on the response process the following formula obtains

\[ \nu^+(\xi; t) = \lim_{\Delta t \to 0} \frac{E[N^+((\xi; t - \Delta t/2, t + \Delta t/2))]}{\Delta t} = \int_0^\infty s f_{X(t)\dot{X}(t)}(\xi, s) \, ds \]

where \( f_{X(t)\dot{X}(t)}(\cdot, \cdot) \) denotes the joint PDF of \( X(t) \) and

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\[ \nu^+(\xi; t) = \int_0^\infty s f_{\dot{X}(t)|X(t)}(s|\xi) \, ds \, f_{X(t)}(\xi) = \mathbb{E}[\dot{X}(t)^+|X(t) = \xi] \, f_{X(t)}(\xi), \]

where \( \dot{X}^+ = \max(\dot{X}, 0). \)
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Note that the parameter of the Poisson distribution is

$$E[N^+(\xi; 0, T)] = \int_0^T \nu^+(\xi; t) \, dt.$$
The Mean Upcrossing Rate

It is expedient to rewrite the extreme value distribution as

\[ F_{M(T)}(\xi) = \text{Prob}(M(T) \leq \xi) = \exp \left\{ -\bar{\nu}^+(\xi) T \right\} , \]

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The averaged mean upcrossing rate \( \bar{\nu}^+(\xi) \) is conveniently estimated from simulated response time histories.
Empirical Estimation of the Mean Upcrossing Rate

\[ n^+(\xi; 0, T) = \text{the counted number of upcrossings during the time interval } (0, T) \text{ from a particular simulated time history.} \]
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The sample mean value estimate of \( \bar{n}^+(\xi) \):

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\hat{\nu}^+(\xi) = \frac{1}{k} \sum_{j=1}^{k} n_j^+(\xi; 0, T)
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For a suitable number \( k \), e.g. \( k \geq 20 - 30 \), a good approximation of the 95% confidence interval for the value \( \bar{\nu}^+(\xi) \) is

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\text{conf. band}(\xi) = \hat{\nu}^+(\xi) \pm 1.96 \frac{\hat{s}(\xi)}{\sqrt{k}}
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The empirical standard deviation \( \hat{s}(\xi) \) is given as

\[
\hat{s}(\xi)^2 = \frac{1}{k-1} \sum_{j=1}^{k} \left( \frac{n_j^+ (\xi; 0, T)}{T} - \hat{\nu}^+ (\xi) \right)^2
\]
The PDF $f_X(x)$ of $X(t)$ is written as

$$f_X(x) = \exp\{-\alpha(x)\},$$

where $\alpha(x)$ is a well-behaved function that is strictly increasing for increasing $x$ for $x \geq x_0$ for some $x_0$. 
Mean Upcrossing Rate versus PDF - stationary case

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- Now we can write

$$\nu^+_X(x) = q \exp \{-\alpha(x) + \delta(x)\},$$

where $q = \mathbb{E}[\dot{X}^+]$, $\exp \{\delta(x)\} = \mathbb{E}[\dot{X}^+|X = x]/\mathbb{E}[\dot{X}^+]$. 

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- $q \exp \{-\alpha(x)\} = q f_X(x)$ expresses the mean upcrossing rate for the case with independent $X(t)$ and $\dot{X}(t)$. 
Assumption: $|\delta(x)|$ is of much slower increase than $\alpha(x)$ as $x \to \infty$. 
Mean Upcrossing Rate versus PDF - stationary case

- Assumption: $|\delta(x)|$ is of much slower increase than $\alpha(x)$ as $x \to \infty$.

- It is seen that

$$\ln \nu_X^+(x) = \ln f_X(x) + \ln q + \delta(x)$$

$$= -\alpha(x) + \ln q + \delta(x)$$
Assumption: $|\delta(x)|$ is of much slower increase than $\alpha(x)$ as $x \to \infty$.

It is seen that

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Plotting $\ln \nu_X^+(x)$ versus $\ln f_X(x)$ will then clearly show to what extent $|\delta(x)|$ is dominated by $\alpha(x)$ as $x \to \infty$. 
Extrapolation of Mean Upcrossing Rate

Assumption:

\[ \alpha(x) = a(x - b)^c - d(x) , \quad x \geq x_0 , \]

where \( a, b \) and \( c \) are suitable constants, and \( d(x) \) is a function of much slower increase than \( \alpha(x) \).
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- Hence, we assume that

\[ \nu_X^+(x) = \tilde{q}(x) \exp\{-a(x - b)^c\}, \quad x \geq x_0, \]

where \( \tilde{q}(x) = q \exp\{\delta(x) + d(x)\} \).
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where \( \tilde{q}(x) = q \exp\{ \delta(x) + d(x) \} \).

The particular choice for the function \( \alpha(x) \) reflects the basic assumption of an asymptotic Gumbel distribution of the extremes.
Extrapolation of Mean Upcrossing Rate

It follows that

$$\log \left| \log \left( \frac{\nu^+_X(x)}{\tilde{q}(x)} \right) \right| = c \log(x - b) + \log a, \ x \geq x_0 (> b).$$
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- Hence, \( \log \left| \log \left( \frac{\nu_X^+(x)}{\tilde{q}(x)} \right) \right| \) plotted versus \( \log(x - b) \) exhibits linear tail behaviour.
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Choice of initial value \( \tilde{q}_0 \) for \( \tilde{q}(x) \) would be based on looking at the ratio \( \nu_X^+(x)/f_X(x) = \tilde{q}(x) \exp\{-d(x)\} \) for large \( x \).
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Practical solution: \( \tilde{q}_0 = \langle \nu_X^+(x)/f_X(x) \rangle \) (tail average), followed by optimization wrt \( b \).
Numerical Examples - Duffing oscillator

The equation of motion is

\[ \ddot{X} + 2\zeta\omega_0 \dot{X} + \omega_0^2 X (1 + \lambda X^2) = W(t) \]
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- \( f_{X\dot{X}} = f_X f_{\dot{X}} \) is known in closed form.
Numerical Examples - Duffing oscillator

Upcrossing rates estimated from Monte Carlo simulations (*) with 95% confidence band (—) versus analytical results (—-—) for the mean upcrossing rate.
Numerical Examples - Duffing oscillator

Monte Carlo (∗) and analytical (—) results the mean upcrossing rate versus PDF on the log scale. Slope = 1.0
Numerical Examples - Hysteretic oscillator

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\( W(t) = \) stationary Gaussian white noise. The hysteretic restoring force is

\[ h(t) = \alpha \omega_0^2 X + (1 - \alpha) Z(t) \]

where \( \alpha = \) post-yielding stiffness parameter (\( 0 \leq \alpha \leq 1 \)).
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- The hysteretic component:

\[ \dot{Z} = -\gamma |\dot{X}| Z |Z|^{\nu-1} - \beta |\dot{X}| Z^\nu + A \dot{X} \]
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- \( \omega_0 = 1, \alpha = 0.05, \zeta = 0.1, A = \nu = 1, \gamma = \beta = 0.5. \)
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- \( \omega_0 = 1, \alpha = 0.05, \zeta = 0.1, A = \nu = 1, \gamma = \beta = 0.5 \).

- \( f_X \dot{X} \) unknown.
Numerical Examples - Hysteretic oscillator

Monte Carlo results for the mean upcrossing rate, 100 realizations (*) along with 95% confidence bands (−−) versus 50000 realizations (—).
Numerical Examples - Hysteretic oscillator

Monte Carlo results for the mean upcrossing rate versus PDF, 100 realizations (*) versus 50000 realizations (—). Slope = 1.02
Numerical Examples - Jacket structure

The Kvitebjørn jacket platform with the superstructure removed.
Numerical Examples - Jacket structure

The equation of motion for the horizontal excursions of the jacket at main deck level is

\[ M \ddot{X} + C \dot{X} + KX = Q. \]
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The equation of motion for the horizontal excursions of the jacket at main deck level is

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{KX} = \mathbf{Q}.$$ 

$$\mathbf{X} = (X_1, \ldots, X_N)^T$$ where \(X_k = X_k(t), \ k = 1, \ldots, N\), denote displacement of the \(k\)-th node \(\mathbf{x}_k = (x_k, y_k, z_k)\) in the wave direction, which is the positive \(x\)-direction.
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- \( Q = (Q(t, x_1), \ldots, Q(t, x_N))^T \), where

\[ Q(t, x_k) = F_{in}(t, x_k) + F_d(t, x_k), \quad k = 1, \ldots, N \]

and

\[ -d = z_1 \leq z_k \leq z_N = L - d, \]

where \( d = 190 \) m is the water depth and \( L = 216 \) m is the jacket support height.
The inertia force components are given as

\[ F_{in}(t, x_k) = k_m \dot{U}(t, x_k) \]
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F_{in}(t, x_k) = k_m \ddot{U}(t, x_k)
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- The drag force components

\[
F_d(t, x_k) = k_d \left( U(t, x_k) + U_c \right) \left| U(t, x_k) + U_c \right|
\]
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$$F_{in}(t, x_k) = k_m \dot{U}(t, x_k)$$

The drag force components

$$F_d(t, x_k) = k_d (U(t, x_k) + U_c) |U(t, x_k) + U_c|$$

$$k_m = C_m \rho \pi D^2 / 4, \quad k_d = C_d \rho D / 2$$
Numerical Examples - Jacket structure

Gumbel plot of 20 simulated 3 hour extremes with fitted Gumbel distribution. Sea state with $H_s = 12$ m, $T_p = 12$ s.
Numerical Examples - Jacket structure

Gumbel plot of 20 simulated 3 hour extremes with fitted Gumbel distribution. Sea state with $H_s = 14.7$ m, $T_p = 15$ s.
Empirical PDF of the 90% fractile value based on samples of size 20 for the sea state with $H_s = 12$ m, $T_p = 12$ s.
Numerical Examples - Jacket structure

Empirical PDF of the 90% fractile value based on samples of size 20 for the sea state with $H_s = 14.7$ m, $T_p = 15$ s.
Numerical Examples - Jacket structure

Mean upcrossing rate statistics along with 95% confidence bands (---) for the sea state with $H_s = 12$ m, $T_p = 12$ s, $\sigma = 0.047$ m. *: Monte Carlo; − − − : linear fit.
Numerical Examples - Jacket structure

Mean upcrossing rate statistics along with 95% confidence bands (---) for the sea state with $H_s = 14.7$ m, $T_p = 15$ s, $\sigma = 0.066$ m. *: Monte Carlo; − − − : linear fit.
Numerical Examples - Jacket structure

Transformed plot along with 95% confidence bands (---) for the sea state with $H_s = 12$ m, $T_p = 12$ s, $\sigma = 0.047$ m. *: Monte Carlo; ------: linear fit, $q = 0.04$, $b = 1.4\sigma$. 

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Numerical Examples - Jacket structure

Transformed plot along with 95% confidence bands (––) for the sea state with $H_s = 14.7$ m, $T_p = 15$ s, $\sigma = 0.066$ m. * : Monte Carlo; ——— : linear fit, $q = 0.06$, $b = 0.9\sigma$. 

![Graph showing the transformed plot with confidence bands and fitting lines for the sea state with given parameters.](image-url)
Numerical Examples - TLP

Sketch of submerged part of TLP.
Numerical Examples - TLP

The equation of motion for the horizontal excursions of the TLP is

\[ M\ddot{Z}(t) + D(t)\dot{Z}(t) + C(\dot{Z}(t)) + K(Z(t)) = F(t) \]
Numerical Examples - TLP

- The equation of motion for the horizontal excursions of the TLP is
  \[ M\ddot{Z}(t) + D(t)\dot{Z}(t) + C(\dot{Z}(t)) + K(Z(t)) = F(t) \]
  
- \[ F(t) = F_1(t) + F_2(t) \]
Numerical Examples - TLP

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- A simplified model for the surge response is adopted here:
  \[ \ddot{Z} + 2 \omega_e (\zeta_0 + \tilde{c}F_2(t)) \dot{Z} + \omega_e^2 (Z + \tilde{\epsilon}Z^3) = \frac{1}{M} (F_1(t) + F_2(t)) \]
Crossing rates by Monte Carlo simulation (*) with 95% confidence bands (— —) and by saddle point integration (——) for the case of linear dynamics ($\tilde{c} = \tilde{\varepsilon} = 0$). Sea state with $H_s = 10$ m, $T_p = 11$ s.
Numerical Examples - TLP

Crossing rates by Monte Carlo simulation (*) with 95% confidence bands (——) and by saddle point integration (—) for the case of linear dynamics ($\tilde{c} = \tilde{\varepsilon} = 0$). Sea state with $H_s = 15 \text{ m}, T_p = 17 \text{ s}$. 

![Graph showing crossing rates with confidence bands and saddle point integration for a sea state with $H_s = 15 \text{ m}, T_p = 17 \text{ s}$]
Numerical Examples - TLP

Crossing rates by Monte Carlo simulation (*) with 95% confidence bands (—) for the case of nonlinear dynamics, \( q = 0.2, b = 5.8 \sigma_Z \).

Sea state with \( H_s = 10 \text{ m}, T_p = 11 \text{ s} \).
Numerical Examples - TLP

Crossing rates by Monte Carlo simulation (*) with 95% confidence bands (––) for the case of nonlinear dynamics, $q = 0.2$, $b = 2.9 \sigma_Z$. Sea state with $H_s = 15 \text{ m}$, $T_p = 17 \text{ s}$. 

nonlinear TLP, sea state with $H_s=15\text{m}$, $T_p=17\text{s}$
Conclusions

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Optimized linear fit and extrapolation on a double logarithmic scale gives accurate predictions of the mean upcrossing rate and thus extreme response statistics.

The CPU time is in all examples tractable, and it is reduced by a factor of $\geq 100$, compared to straight-forward Monte Carlo simulations down to the same extreme value levels.